

Feynman-Kac particle integration with geometric interacting jumps

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Abstract

This article is concerned with the design and analysis of discrete time Feynman-Kac particle integration models with geometric interacting jump processes. We analyze two general types of model, corresponding to whether the reference process is in continuous or discrete time. For the former, we consider discrete generation particle models defined by arbitrarily fine time mesh approximations of the Feynman-Kac models with continuous time path integrals. For the latter, we assume that the discrete process is observed at integer times and we design new approximation models with geometric interacting jumps in terms of a sequence of intermediate time steps between the integers. In both situations, we provide non asymptotic bias and variance theorems w.r.t. the time step and the size of the system, yielding what appear to be the first results of this type for this class of Feynman-Kac particle integration models. We also discuss uniform convergence estimates w.r.t. the time horizon. Our approach is based on an original semigroup analysis with first order decompositions of the fluctuation errors.

Keywords : Feynman-Kac formulae, interacting jump particle systems, measure valued processes, non asymptotic bias and variance estimates.

Mathematics Subject Classification :

Primary: 62L20; 65C05; 60G35. ; Secondary: 60G57, 81Q05; 82C22.

1 Introduction

Feynman-Kac formulae are central path integration mathematical models in physics and probability theory. More precisely, these models and their interacting particle interpretations have come to play a significant role in applied probability, numerical physics, Bayesian statistics, probabilistic machine learning, and engineering sciences. Applications of these particle integration techniques are increasingly used to solve a variety of complex problems in nonlinear filtering, data assimilation, rare event sampling, hidden Markov chain parameter estimation, stochastic control and financial mathematics. A detailed account of these functional models and their application domains can be found in the series of research books [3, 10, 21, 27, 34] and, more recently, in [4, 16, 26].

In computational physics, these techniques are used for free energy computations, specifically in estimating ground states of Schrödinger operators. In this context, these particle models are often referred as quantum or diffusion Monte Carlo methods [1, 2, 7, 39]. We also refer the reader to the series of articles [11, 24, 40, 41, 46].

In advanced signal processing, they are known as particle filters or sequential Monte Carlo methods, and were introduced in three independent works in the 90's [9, 32, 37]. These stochastic particle algorithms are now routinely used to compute sequentially the flow of conditional distributions of the random states of a signal process given some noisy and partial observations [3, 10, 12, 21, 27, 28, 35, 38]. Feynman-Kac formulae and their particle interpretations are also commonly used in financial mathematics to model option prices, futures prices and sensitivity measures, and in insurance and risk models [4, 5, 33, 42, 44, 43]. They are used in rare event analysis to model conditional distributions of stochastic processes evolving in a rare event regime [6, 5, 20].

This article presents geometric interacting jump particle approximations of Feynman-Kac path integrals. It also contains theoretical results related to the practical implementation of these particle algorithms for both discrete and continuous time integration problems. A key result is the presentation of connections between the interacting jump particle interpretations of the continuous time models and their discrete time generation versions. This is motivated by the fact that while the continuous time nature of these models is fundamental to describing certain phenomena, the practical implementation of these models on a computer requires a judicious choice of time discretization. Conversely, as shown in section 2.1 in [25], a discrete time Feynman-Kac model can be encapsulated within a continuous time framework by considering stochastic processes only varying on integer times. Continuous time Feynman-Kac particle models are based on exponential interacting jumps [15, 24, 21, 30, 31, 29, 46], while their discrete time versions are based on geometric type jumps [10, 16, 19]. From a computational perspective, the exponential type interacting jumps thus need to be approximated by geometric type jumps. Incidentally, some of these geometric type interacting jump particle algorithms are better suited to implementation in a parallel computing environment (see section 5.3).

Surprisingly, little attention has been paid to analyze the connections between exponential and geometric type jump particle models. There are references dealing with these two models separately [8, 17, 18, 22, 25, 24, 46], but none provide a convergence analysis between the two. In this paper we initiate this study with a non asymptotic bias and variance analysis w.r.t. the time step parameter and the size of the particle population scheme. Special attention is paid to the stochastic modeling of these interacting jump processes, and to a stochastic perturbation analysis of these particle models w.r.t. local sampling random fields.

We conclude this section with basic notation used in the article. We let $\mathcal{B}_b(E)$ be the Banach space of all bounded Borel functions f on some Polish¹ state space E equipped with a Borel σ -field \mathcal{E} , equipped with the uniform norm $\|f\| = \sup_{x \in E} |f(x)|$. We denote by $\text{osc}(f) := \sup_{x,y} |f(x) - f(y)|$ the oscillation of a function $f \in \mathcal{B}_b(E)$. We let $\mu(f) = \int f(x)\mu(dx)$ be the Lebesgue integral of a function $f \in \mathcal{B}_b(E)$ with respect to a finite signed measure μ on E . We also equip the set $\mathcal{M}(E)$ of finite signed measures μ with the total variation norm $\|\mu\|_{\text{tv}} = \sup |\mu(f)|$, where the supremum is taken over all functions $f \in \mathcal{B}_b(E)$ with $\text{osc}(f) \leq 1$. We let $\mathcal{P}(E) \subset \mathcal{M}(E)$ be the subset of all probability measures. We recall that any bounded integral operator Q on E is an operator Q from $\mathcal{B}_b(E)$ into itself defined by $Q(f)(x) = \int Q(x, dy)f(y)$, for some measure $Q(x, \cdot)$, indexed by $x \in E$, and we set

¹i.e. homeomorphic to a complete separable metric space

$\|Q\|_{\text{tv}} = \sup_{x \in E} \|Q(x, \cdot)\|_{\text{tv}}$. These operators generate a dual operator $\mu \mapsto \mu Q$ on the set of finite signed measures defined by $(\mu Q)(f) = \mu(Q(f))$. A Markov kernel is a positive and bounded integral operator Q s.t. $Q(1) = 1$. The Dobrushin contraction coefficient of a Markov kernel Q is defined by $\beta(Q) := \sup \text{osc}(Q(f))$, where the supremum is taken over all functions $f \in \mathcal{B}_b(E)$ s.t. $\text{osc}(f) \leq 1$. Given some positive potential function G on E , we denote by Ψ_G the Boltzmann-Gibbs transformation $\mu \in \mathcal{P}(E) \mapsto \Psi_G(\mu) \in \mathcal{P}(E)$ defined by $\Psi_G(\mu)(f) = \mu(fG)/\mu(G)$.

2 Description of the models

2.1 Feynman-Kac models

We consider an E -valued Markov process \mathcal{X}_t , $t \in \mathbb{R}_+ = [0, \infty[$ defined on a standard filtered probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$. The set $\Omega = D(\mathbb{R}_+, E)$ represents the space of càdlàg paths equipped with the Skorokhod topology which turn it into a Polish space. A point $\omega \in \Omega$ represents a sample path of the canonical process $\mathcal{X}_t(\omega) = \omega_t$. We also let $\mathcal{F}_t^X = \sigma(\mathcal{X}_s, s \leq t)$ and \mathbb{P} be the sigma-field and probability measure of the process $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$. Finally, we also consider the \mathbb{P} -augmentation \mathcal{F}_t of \mathcal{F}_t^X so that the resulting filtration satisfies the usual conditions of right continuity and completion by \mathbb{P} -negligible sets (see for instance [36, 45], and the references therein). We also consider a time inhomogeneous bounded Borel function \mathcal{V}_t on E .

We let \mathbb{Q}_t and Λ_t be the Feynman-Kac measures on $\Omega_t := D([0, t], E)$ defined for any bounded measurable function f on Ω_t , by the following formulae

$$\mathbb{Q}_t(f) := \Lambda_t(f)/\Lambda_t(1) \quad \text{with} \quad \Lambda_t(f) = \mathbb{E} \left(f(\mathcal{X}_{[0,t]}) \exp \left(\int_0^t \mathcal{V}_s(\mathcal{X}_s) ds \right) \right) \quad (1)$$

and we let ν_t and μ_t , respectively, be the t -marginals of Λ_t and \mathbb{Q}_t .

We consider the mesh sequence $t_k = k/m$, $k \geq 0$, with time step $h = t_n - t_{n-1} = 1/m$ associated with some integer $m \geq 1$, and we let $\mathbb{Q}_{t_n}^{(m)}$ and $\Lambda_{t_n}^{(m)}$ be the Feynman-Kac measures on Ω_{t_n} defined for any bounded measurable function f on Ω_{t_n} , by the following formulae

$$\mathbb{Q}_{t_n}^{(m)}(f) := \Lambda_{t_n}^{(m)}(f)/\Lambda_{t_n}^{(m)}(1) \quad \text{with} \quad \Lambda_{t_n}^{(m)}(f) = \mathbb{E} \left(f(\mathcal{X}_{[0,t_n]}) \prod_{0 \leq p < n} e^{\mathcal{V}_{t_p}(\mathcal{X}_{t_p})/m} \right). \quad (2)$$

We also denote by $\nu_{t_n}^{(m)}$ and $\mu_{t_n}^{(m)}$, respectively, the t_n -marginal of $\Lambda_{t_n}^{(m)}$ and $\mathbb{Q}_{t_n}^{(m)}$.

- **Case (D) :** We have $\mathcal{X}_t = X_{[t]}$ and $\mathcal{V}_t = \log G_{[t]}$, where X_n , $n \in \mathbb{N}$ is an E -valued Markov chain, and G_n are Borel positive functions s.t. $\log G_n$ is bounded.

In this case, the marginal $\nu_n = \gamma_n$ and $\mu_n = \eta_n$ of the Feynman-Kac measures of Λ_t and \mathbb{Q}_t on integer times $t = n$ are given for any $f \in \mathcal{B}_b(E)$ by the formula

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E} \left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right). \quad (3)$$

- **Case (C) :** The process \mathcal{X}_t is a continuous time Markov process with infinitesimal generators $L_t : D(L) \rightarrow D(L)$ defined on some common domain of functions $D(L)$, and $\mathcal{V} \in \mathcal{C}^1(\mathbb{R}_+, D(L))$. The set $D(L)$ is a sub-algebra of the Banach space $\mathcal{B}_b(E)$ generating the Borel σ -field \mathcal{E} , and for any measurable function $\mathcal{U} : t \in \mathbb{R}_+ \mapsto \mathcal{U}_t \in D(L)$ the Feynman-Kac semigroup $\mathcal{Q}_{s,t}$, $s \leq t$, defined by

$$\mathcal{Q}_{s,t}(f)(x) = \mathbb{E} \left(f(\mathcal{X}_t) \exp \left(\int_s^t \mathcal{U}_r(\mathcal{X}_r) dr \right) \mid \mathcal{X}_s = x \right)$$

leaves $D(L)$ invariant; that is we have that $\mathcal{Q}_{s,t}(D(L)) \subset D(L)$. For any $s \leq t$, the mappings $r \in [0, s] \mapsto L_r(Q_{s,t}(f)^2)$ and $r \in [0, s] \mapsto L_r^2(Q_{s,t}(f)^2) \in \mathcal{C}^1([0, s], D(L))$, and their norm as well the norm of the first order derivatives only depend on $(\mathcal{X}_s)_{s \leq t}$ and on the norms of the functions $(\mathcal{U}_s)_{s \leq t}$ and their derivatives.

The regularity conditions stated in **(C)** correspond to time inhomogeneous versions of those introduced in [24]. They hold for pure jump processes with bounded jump rates with $D(L) = \mathcal{B}_b(E)$, or for Euclidean diffusions on $E = \mathbb{R}^d$ with regular and Lipschitz coefficients by taking $D(L)$ as the set of \mathcal{C}^∞ -functions with derivatives decreasing at infinity faster than any polynomial function. These regularity conditions allow the use of most of the principal theorems of stochastic differential calculus, e.g. the “carré du champ”, or square field, operator that characterizes the predictable quadratic variations of the martingales that appear in Ito’s formulae. These regularity conditions can probably be relaxed using the extended setup developed in [21].

We have already mentioned that the particle interpretations associated with the continuous time models (1) are defined in terms of interacting jump particle systems [21, 22, 24, 25]. The implementation of these continuous time particle algorithms is clearly impractical and we therefore resort to the geometric interacting processes associated with the m -approximation models defined in (2). These discrete generation interacting jumps models provide new and different types of adaptive resampling procedures, which differ from those discussed in the articles [12, 13], and the references therein.

2.2 Mean field particle models

In this section, we provide a brief description of the geometric type interacting jump particle models associated with the m -approximation Feynman-Kac model defined in (2). First, if we define

$$\mathcal{M}_{t_n, t_{n+1}}(x, dy) = \mathbb{P}(\mathcal{X}_{t_{n+1}} \in dy \mid \mathcal{X}_{t_n} = x) \quad \text{and} \quad \mathcal{G}_{t_n} = \exp(\mathcal{V}_{t_n}/m),$$

then it is well known that $\mu_{t_n}^{(m)}$ satisfies the following evolution equation

$$\mu_{t_{n+1}}^{(m)} = \Psi_{\mathcal{G}_{t_n}}(\mu_{t_n}^{(m)}) \mathcal{M}_{t_n, t_{n+1}}. \quad (4)$$

Further details on the derivation of these evolution equations can be found in [10, 21, 16]. The particle interpretation of this model depends on the interpretation of the Boltzmann-Gibbs transformation in terms of a Markov transport equation

$$\Psi_{\mathcal{G}_{t_n}}(\mu) = \mu \mathcal{S}_{t_n, \mu} \quad (5)$$

for some Markov transitions $\mathcal{S}_{t_n, \mu}$, that depend on the time parameter t_n and on the measure μ . The choice of these Markov operators is not unique; we refer to [10] for a more thorough discussion of

these models. In this article, we consider an abstract general model, and illustrate our study with the following three classes of models.

- **Case 1 :** We have $\mathcal{V}_t = -\mathcal{U}_t$, for some non negative and bounded function \mathcal{U}_t . In this situation, (5) is satisfied by the Markov transition

$$\mathcal{S}_{t_n, \mu}(x, dy) := e^{-\mathcal{U}_{t_n}(x)/m} \delta_x(dy) + \left(1 - e^{-\mathcal{U}_{t_n}(x)/m}\right) \Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu)}(dy).$$

- **Case 2 :** The function \mathcal{V}_t is non negative. In this situation, (5) is satisfied by the Markov transition

$$\mathcal{S}_{t_n, \mu}(x, dy) := \frac{1}{\mu(e^{\mathcal{V}_{t_n}/m})} \delta_x(dy) + \left(1 - \frac{1}{\mu(e^{\mathcal{V}_{t_n}/m})}\right) \Psi_{(e^{\mathcal{V}_{t_n}/m}-1)(\mu)}(dy).$$

- **Case 3:** The Markov transport equation (5) is satisfied by the Markov transition

$$\mathcal{S}_{t_n, \mu}(x, dy) := (1 - a_{t_n, \mu}(x)) \delta_x(dy) + a_{t_n, \mu}(x) \Psi_{(e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+}(\mu)(dy)$$

with the rejection rate $a_{t_n, \mu}(x) := \mu\left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+\right) / \mu(e^{\mathcal{V}_{t_n}/m}) \in [0, 1]$

In these three cases we have the following first order expansion

$$\mathcal{S}_{t_n, \mu} = Id + \frac{1}{m} \hat{L}_{t_n, \mu} + \frac{1}{m^2} \hat{R}_{t_n, \mu} \quad (6)$$

with some jump type generator $\hat{L}_{t_n, \mu}$ and some integral operator $\hat{R}_{t_n, \mu}^{(m)}$ s.t. $\sup \left\| \hat{R}_{t_n, \mu}^{(m)} \right\|_{\text{tv}} < \infty$, where the supremum is taken over all $m \geq 1$ and $\mu \in \mathcal{P}(E)$. The jump generators $\hat{L}_{t_n, \mu}$ corresponding to the three cases presented above are described respectively in (12), (13), and (14). The proofs of these expansions is rather elementary, and they are housed in the appendix, on page 37.

In addition, whenever (5) is satisfied, we have the evolution equation

$$\mu_{t_{n+1}}^{(m)} = \mu_{t_n}^{(m)} \mathcal{K}_{n+1, \mu_{t_n}^{(m)}} \quad \text{with the Markov kernels} \quad \mathcal{K}_{t_n, t_{n+1}, \mu} = \mathcal{S}_{t_n, \mu} \mathcal{M}_{t_n, t_{n+1}}. \quad (7)$$

The mean field N -particle model $\xi_{t_n} := (\xi_{t_n}^i)_{1 \leq i \leq N}$ associated with the evolution equation (7) is a Markov process in E^N with elementary transitions given by

$$\mathbb{P}(\xi_{t_{n+1}} \in dx \mid \xi_{t_n}) = \prod_{1 \leq i \leq N} \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(\xi_{t_n}^i, dx^i) \quad \text{with} \quad \mu_{t_n}^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_{t_n}^i}, \quad (8)$$

where $dx = dx^1 \dots dx^N$ stands for an infinitesimal neighborhood of the point $x = (x^i)_{1 \leq i \leq N} \in E^N$.

3 Statement of the main results

Our first main result relates the Feynman-Kac models (1) and their m -approximation measures (2) in case **(D)** and **(C)**.

Theorem 3.1 *In case **(D)**, we have*

$$\nu_n^{(m)} = \nu_n = \gamma_n \quad \text{and} \quad \mu_n^{(m)} = \mu_n = \eta_n$$

with the Feynman-Kac measures γ_n and η_n defined in (3).

*In case **(C)**, we have the first order decomposition*

$$\Lambda_{t_n}^{(m)} = \Lambda_{t_n} + \frac{1}{m} r_{m,t_n} \quad \text{and} \quad \mathbb{Q}_{t_n}^{(m)} = \mathbb{Q}_{t_n} + \frac{1}{m} \bar{r}_{m,t_n}$$

with some remainder signed measures $r_{m,t_n}, \bar{r}_{m,t_n}$ s.t. $\sup_{m \geq 1} [\|\bar{r}_{m,t_n}\|_{\text{tv}} \vee \|r_{m,t_n}\|_{\text{tv}}] < \infty$.

The proof of the theorem is rather technical and it is postponed to the appendix.

The first assertion of theorem 3.1 allows us to turn a discrete time Feynman-Kac model (3) into a continuous time model (1). To be more precise, we have that $\nu_{t_n}^{(m)} = \nu_{t_p}^{(m)} Q_{t_p, t_n}^{(m)}$ with the Feynman-Kac semigroup

$$Q_{t_p, t_n}^{(m)}(f)(x) := \mathbb{E} \left(f(\mathcal{X}_{t_n}) \prod_{p \leq q < n} e^{\mathcal{V}_{t_q}(\mathcal{X}_{t_q})/m} \mid \mathcal{X}_{t_p} = x \right)$$

in case **(D)**, for integer times $(t_p, t_n) = (km, nm)$, with $k \leq n$, we also have that $\gamma_n = \gamma_k Q_{k,n}$ with the Feynman-Kac semigroup

$$Q_{k,n}(f)(x) := \mathbb{E} \left(f(X_n) \prod_{k \leq l < n} G_l(X_l) \mid X_k = x \right) = Q_{k,n}^{(m)}(f)(x).$$

Thus, the normalized Markov kernels $P_{k,n}^{(m)}(f) := Q_{k,n}^{(m)}(f)/Q_{k,n}^{(m)}(1)$ also coincide with the Markov kernels $P_{k,n}(f) := Q_{k,n}(f)/Q_{k,n}(1)$. In addition, for any $k \geq 0$ and $r < m$, we also have the semigroup formulae

$$Q_{k,n}^{(m)}(f)(x) = G_k(x)^{r/m} Q_{k+r/m,n}^{(m)}(f)(x) \quad \text{and} \quad P_{k,n}^{(m)} = P_{k+r/m,n}^{(m)} = P_{k,n}. \quad (9)$$

We prove the l.h.s. assertion using the fact that for any $n \geq 0$ and any $p = km + r$, with $k \geq 0$ and $r < m$, we have $t_p = k + r/m$ and

$$Q_{k,n}^{(m)}(f)(x) = G_k(x)^{r/m} \times \mathbb{E} \left(f(\mathcal{X}_{t_{nm}}) \prod_{k+r/m \leq t_q < n} e^{\mathcal{V}_{t_q}(\mathcal{X}_{t_q})/m} \mid \mathcal{X}_{k+r/m} = x \right).$$

For a Feynman-Kac measure (1) associated with a continuous diffusion style process \mathcal{X}_t , it is important to observe that the l.h.s. measure in the m -approximation model (2), as defined on a time mesh sequence, can be thought of as a time discretization of the exponential path integrals in the continuous time model (1). Nevertheless, the elementary Markov transitions of the Markov chain

$(\mathcal{X}_{t_n})_{n \geq 0}$ are generally unknown. To get some feasible Monte Carlo approximation scheme, we need a dedicated technique to sample the transitions of this chain. One natural strategy is to replace in (2), the reference Markov chain $(\mathcal{X}_{t_n})_{n \geq 0}$ by the Markov chain $(\hat{\mathcal{X}}_{t_n})_{n \geq 0}$ associated with some Euler type discretization model with time step $\Delta t = 1/m$. The stochastic analysis of these models is discussed in some detail in the articles [17, 18, 15], including first order expansions in terms of the size of the time mesh sequence.

Our second main result is the following non asymptotic bias and variance theorem for the N -approximation mean field model introduced in (8).

Theorem 3.2 *We assume that the Markov transport equation (5) is satisfied for Markov transitions $\mathcal{S}_{t_n, \mu}$ also satisfying the first order decomposition (6).*

In case (C), for any function $f \in D(L)$, and any $N \geq m \geq 1$ we have the non asymptotic bias and variance estimates

$$|\mathbb{E}(\mu_{t_n}^N(f)) - \mu_{t_n}(f)| \leq c_{t_n}(f) \left[\frac{1}{N} + \frac{1}{m} \right]$$

and

$$\mathbb{E}([\mu_{t_n}^N(f) - \mu_{t_n}(f)]^2) \leq c_{t_n}(f) \left[\frac{1}{N} + \frac{1}{m^2} \right]$$

for some finite constant $c_{t_n}(f) < \infty$ that only depends on t_n and on f .

In case (D), for any $f \in \mathcal{B}_b(E)$ s.t. $\text{osc}(f) \leq 1$, and for any $N \geq m \geq 1$ we have the non asymptotic bias and variance estimates

$$N |\mathbb{E}(\mu_n^N(f)) - \eta_n(f)| \leq a(n) \quad \text{and} \quad N \mathbb{E}([\mu_n^N(f) - \mu_{t_n}(f)]^2) \leq a(n) \left(1 + \frac{1}{N} a(n) \right)$$

for some constant

$$a(n) \leq c \sum_{0 \leq k < n} g_{k,n}^3 g_{k,k+1}^3 (\|\log G_k\| \vee 1)^2 \beta(P_{k,n}) \quad \text{with} \quad g_{k,n} := \sup_{x,y} Q_{k,n}(1)(x)/Q_{k,n}(1)(y).$$

Under appropriate regularity conditions on the Feynman-Kac model, we can prove that the constant $a(n)$ is uniformly bounded w.r.t. the time parameter; that is we have that $\sup_{n \geq 0} a(n) < \infty$. For a detailed discussion of these uniform convergence properties w.r.t. the time parameter, we refer the reader to the book [10], and the more recent article [16]. To be more precise, we let $\Phi_{k,l}(\eta_k) = \eta_l$ be the Feynman-Kac semigroup associated with the flow of measures η_k . In this notation, by proposition 2.3 in [23] we have that the Dobrushin contraction coefficient of the Markov kernel $Q_{k,n}(f)/Q_{k,n}(1)$ is given by

$$\beta(P_{k,n}) = \sup_{\mu_1, \mu_2 \in \mathcal{P}(E)} \|\Phi_{k,n}(\mu_1) - \Phi_{k,n}(\mu_2)\|_{\text{tv}}.$$

On the other hand, we also have that

$$Q_{k,n}(1)(x) = \prod_{k \leq l < n} \Phi_{k,l}(\delta_x)(G_l) \Rightarrow \log \frac{Q_{k,n}(1)(x)}{Q_{k,n}(1)(y)} = \sum_{k \leq l < n} (\log \Phi_{k,l}(\delta_x)(G_l) - \log \Phi_{k,l}(\delta_y)(G_l)).$$

Using the fact that $\log a - \log b = \int_0^1 \frac{(a-b)}{ta+(1-t)b} dt$, we find that

$$\log \frac{Q_{k,n}(1)(x)}{Q_{k,n}(1)(y)} = \sum_{k \leq l < n} \int_0^1 \frac{[\Phi_{k,l}(\delta_x)(G_l) - \Phi_{k,l}(\delta_y)(G_l)]}{t\Phi_{k,l}(\delta_x)(G_l) + (1-t)\Phi_{k,l}(\delta_y)(G_l)} dt.$$

Assuming that for any l and x , and

$$c_1 \leq G_l(x) \leq c_2 \quad \text{and} \quad \sup_{\mu_1, \mu_2 \in \mathcal{P}(E)} \|\Phi_{k,l}(\mu_1) - \Phi_{k,l}(\mu_2)\|_{\text{tv}} \leq c_3 e^{-c_4(k-l)} \quad (10)$$

for some positive and bounded constants c_i , $1 \leq i \leq 4$, we find that

$$\beta(P_{k,n}) \leq c_3 e^{-c_4(k-n)} \quad \text{and} \quad \log g_{k,n} \leq 2(c_2 c_3 / c_1) \left(\sum_{k \leq l < n} e^{-c_4(k-l)} \right) \leq 2(c_2 c_3) / (c_1 (1 - e^{-c_4})).$$

This clearly implies that $(10) \Rightarrow \sup_{n \geq 0} a(n) < \infty$.

For instance, it was proven in [21, 14] that condition (10) is met for time homogeneous models as soon as the Markov transition M of the Markov chain X_n satisfies the following mixing condition

$$\exists m \geq 1, \exists \rho > 0 : \forall x, y \in E \quad M^m(x, \cdot) \geq \rho M^m(y, \cdot).$$

It is well known that this condition is satisfied for any aperiodic and irreducible Markov chains on finite state spaces, as well as for bi-Laplace exponential transitions associated with a bounded drift function, and for Gaussian transitions with a mean drift function that is constant outside some compact domain.

The remainder of the article is organized as follows:

Section 4 is concerned with continuous time particle interpretations of the Feynman-Kac models (1). By the representation theorem 3.1, these schemes also provide a continuous time particle interpretation of the discrete time models (3) without further work. In section 4.2, we present the McKean interpretation of the Feynman-Kac models in terms of a time inhomogeneous Markov process whose generator depends on the distribution of the random states. The choice of these McKean models is not unique. We discuss the three interpretation models corresponding to the three selection type transitions presented on page 5. The mean field particle interpretation of these McKean models are discussed in section 4.3.

Of course, even for discrete time models (3) these continuous time particle interpretations are based on continuous time interacting jump models and they cannot be used in practice without an additional level of approximation. In this context, when using an Euler type approximation these exponential interacting jumps are replaced by geometric type recycling clocks. These interacting geometric jumps particle models are discussed in section 5, which is dedicated to the discrete time particle interpretations of the Feynman-Kac models presented in (2). In section 5.1, we discuss the McKean interpretation of the Feynman-Kac models in terms of a time inhomogeneous Markov chain model whose elementary transitions depends on the distribution of the random states. Again, the choice of these McKean models is not unique. We discuss the three interpretation models corresponding to the three cases presented on page 5. The mean field particle interpretation of these McKean models are discussed on page 20.

Once again, using the representation formulae (2) we emphasize that these schemes also provide a discrete generation particle interpretation of the discrete time models (3). In contrast to standard discrete generation particle models associated with (3), these particle schemes are defined on a refined time mesh sequence between integers. This time mesh sequence can be interpreted as a time dilation. Between two integers, the particle evolution undergoes an additional series of intermediate time evolution steps. In each of these time steps, a dedicated Bernoulli acceptance-rejection trial coupled with a recycling scheme is performed. As the time step decreases to 0, the resulting geometric interacting jump processes converge to the exponential interacting jump processes associated with the continuous time particle model. The final section, section 6, is mainly concerned with the proof of theorem 3.2.

4 Continuous time models

4.1 Feynman-Kac semigroups

In case (C) the semigroup of the flow of non negative measures ν_t is given for any $s \leq t$ by the following formulae $\nu_t = \nu_s Q_{s,t}$, with the Feynman-Kac semigroup $Q_{s,t}$ defined for any $f \in \mathcal{B}(E)$ by

$$Q_{s,t}(f)(x) = \mathbb{E} \left(f(\mathcal{X}_t) \exp \left\{ \int_s^t \mathcal{V}_s(\mathcal{X}_s) ds \right\} \mid X_s = x \right).$$

This yields $\mu_t = \Phi_{s,t}(\mu_s)$, with the nonlinear transformation $\Phi_{s,t}$ on the set of probability measures defined for any $f \in \mathcal{B}(E)$ by

$$\Phi_{s,t}(\mu_s)(f) := \mu_s(Q_{s,t}(f)) / \mu_s(Q_{s,t}(1)).$$

Using some stochastic calculus manipulations, we readily prove that μ_t satisfies the following integro-differential equation

$$\frac{d}{dt} \mu_t(f) = \mu_t(L_t(f)) + \mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t) \mu_t(f) \quad (11)$$

for any function $f \in D(L)$. Further details on the derivation of these evolution equations can be found in the articles [24, 22]. The particle interpretation of this model depends on the interpretation of the correlation term in the r.h.s. of (11) in terms of a jump type generator. The choice of these generators is not unique. Next, we discuss three important classes of models. These three situations are the continuous time versions of the three cases discussed on page 5.

- **Case 1 :** We assume that $\mathcal{V}_t = -\mathcal{U}_t$, for some non negative function \mathcal{U}_t . In this situation, we have the formula

$$\mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t) \mu_t(f) = \mu_t(\mathcal{U}_t [\mu_t(f) - f]) = \mu_t(\widehat{L}_{t,\mu_t}(f))$$

with the interacting jump generator

$$\widehat{L}_{t,\mu_t}(f)(x) = \mathcal{U}_t(x) \int [f(y) - f(x)] \mu_t(dy). \quad (12)$$

- **Case 2 :** When \mathcal{V}_t is a positive function, then we have the formula

$$\mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t)\mu_t(f) = \mu_t\left(\widehat{L}_{t,\mu_t}(f)\right)$$

with the interacting jump generator

$$\widehat{L}_{t,\mu_t}(f)(x) = \int [f(y) - f(x)] \mathcal{V}_t(y) \mu_t(dy) = \mu_t(\mathcal{V}_t) \int [f(y) - f(x)] \Psi_{\mathcal{V}_t}(\mu_t)(dy). \quad (13)$$

- **Case 3 :** For any bounded potential function \mathcal{V}_t we have

$$\mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t)\mu_t(f) = \int (f(y) - f(x)) (\mathcal{V}_t(y) - \mathcal{V}_t(x))_+ \mu_t(dx) \mu_t(dy) = \mu_t\left(\widehat{L}_{t,\mu_t}(f)\right)$$

with $a_+ = a \vee 0$, and with the interacting jump generator

$$\widehat{L}_{t,\mu_t}(f)(x) = \int [f(y) - f(x)] (\mathcal{V}_t(y) - \mathcal{V}_t(x))_+ \mu_t(dy). \quad (14)$$

4.2 McKean interpretation models

In the three cases discussed above, for any test functions $f \in D(L)$ we have the evolution equation

$$\frac{d}{dt}\mu_t(f) = \mu_t(L_{t,\mu_t}(f)) \quad \text{with} \quad L_{t,\mu_t} := L_t + \widehat{L}_{t,\mu_t}. \quad (15)$$

These integro-differential equations can be interpreted as the evolution of the laws, given by $\text{Law}(\overline{X}_t) = \mu_t$, of a time inhomogeneous Markov process \overline{X}_t with infinitesimal generators L_{t,μ_t} that depend on the distribution of the random states at the previous time increment. This probabilistic model is called the McKean interpretation of the evolution equation (15) in terms of a time inhomogeneous Markov process. In this framework, using Ito's formula for any test function $f \in \mathcal{C}^1([0, \infty[, \mathcal{D}(L))$, we have

$$df_t(\overline{\mathcal{X}}_t) = \left(\frac{\partial}{\partial t} + L_{t,\mu_t} \right) (f_t)(\overline{\mathcal{X}}_t) + d\overline{M}_t(f) \quad (16)$$

with a martingale term $\overline{M}_t(f)$ with predictable angle bracket

$$d\langle \overline{M}(f) \rangle_t = \Gamma_{L_{t,\mu_t}}(f_t, f_t)(\overline{\mathcal{X}}_t) dt.$$

Using the r.h.s. description of L_{t,μ_t} in (15), for any $f \in D(L)$ we notice that

$$\Gamma_{L_{t,\mu_t}}(f, f) = \Gamma_{L_t}(f, f) + \Gamma_{\widehat{L}_{t,\mu_t}}(f, f).$$

Next, we provide a description of this Markov process in the three cases discussed above.

- **Case 1:** In this situation, between the jump times the process $\overline{\mathcal{X}}_t$ evolves as the process \mathcal{X}_t . The rate of the jumps is given by the function \mathcal{U}_t . In other words, the jump times $(T_n)_{n \geq 0}$ are given by the following recursive formulae

$$T_{n+1} = \inf \left\{ t \geq T_n : \int_{T_n}^t \mathcal{U}_s(\overline{\mathcal{X}}_s) ds \geq e_n \right\}$$

where $T_0 = 0$, and $(e_n)_{n \geq 0}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n the process $\overline{\mathcal{X}}_{T_n-} = x$ jumps to new site $\overline{\mathcal{X}}_{T_n} = y$ randomly chosen with the distribution $\mu_{T_n-}(dy)$.

For any $f \in D(L)$ we also have that

$$\Gamma_{\widehat{L}_{t,\mu_t}}(f, f)(x) = \widehat{L}_{t,\mu_t}((f - f(x))^2)(x) = \mathcal{U}_t(x) \int [f(y) - f(x)]^2 \mu_t(dy).$$

In this situation, an explicit expression of the time inhomogeneous semigroup $\overline{\mathcal{P}}_{s,t,\mu_s}$, $s \leq t$, of the process $\overline{\mathcal{X}}_t$ is provided by the following formula

$$\begin{aligned} \overline{\mathcal{P}}_{s,t,\mu_s}(f)(x) &= \mathbb{E}(f(\overline{\mathcal{X}}_t) \mid \overline{\mathcal{X}}_s = x) \\ &= Q_{s,t}(1)(x) \Phi_{s,t}(\delta_x)(f) + (1 - Q_{s,t}(1)(x)) \Phi_{s,t}(\mu_s)(f) \\ &= Q_{s,t}(f)(x) + (1 - Q_{s,t}(1)(x)) \Phi_{s,t}(\mu_s)(f). \end{aligned}$$

We let $\overline{\mathcal{P}}_{t_n,t_{n+1},\mu_{t_n}}^{(m)}$ and $\Phi_{t_n,t_{n+1}}^{(m)}$ be the Markov transition and the transformation of probability measures defined as $\overline{\mathcal{P}}_{s,t,\mu_s}$ and $\Phi_{t_n,t_{n+1}}$ replacing $Q_{t_n,t_{n+1}}$ by the integral operator

$$Q_{t_n,t_{n+1}}^{(m)}(f)(x) = e^{-\mathcal{U}_{t_n}(x)/m} \mathbb{E}(f(\mathcal{X}_{t_{n+1}}) \mid \mathcal{X}_{t_n} = x).$$

Under the assumptions of theorem 3.1, using elementary calculations we prove that

$$\overline{\mathcal{P}}_{t_n,t_{n+1},\mu_{t_n}}^{(m)} = \overline{\mathcal{P}}_{t_n,t_{n+1},\mu_{t_n}} + \frac{1}{m} \mathcal{R}_{t_n,t_{n+1},\mu_{t_n}}^{(m)} \quad (17)$$

with some remainder signed measures $\mathcal{R}_{t_n,t_{n+1},\mu_{t_n}}^{(m)}$ such that $\sup_{m \geq 1} \left\| \mathcal{R}_{t_n,t_{n+1},\mu_{t_n}}^{(m)} \right\|_{\text{tv}} \leq c_{t_n}$, for some finite constant whose values only depend on the potential function \mathcal{U}_t .

- **Case 2:** In this situation, between jump times the process $\overline{\mathcal{X}}_t$ evolves as the process \mathcal{X}_t . The rate of the jumps is given by the parameter $\mu_t(\mathcal{V}_t)$. In other words, the jump times $(T_n)_{n \geq 0}$ are given by the following recursive formulae

$$T_{n+1} = \inf \left\{ t \geq T_n : \int_{T_n}^t \mu_s(\mathcal{V}_s) ds \geq e_n \right\}$$

where $T_0 = 0$, and $(e_n)_{n \geq 0}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n the process $\overline{\mathcal{X}}_{T_n-} = x$ jumps to new site $\overline{\mathcal{X}}_{T_n} = y$ randomly chosen with the distribution $\Psi_{\mathcal{V}_{T_n-}}(\mu_{T_n-})(dy)$.

For any $f \in D(L)$ we also have that

$$\Gamma_{\widehat{L}_{t,\mu_t}}(f, f)(x) = \widehat{L}_{t,\mu_t}((f - f(x))^2)(x) = \int [f(y) - f(x)]^2 \mathcal{V}_t(y) \mu_t(dy).$$

- **Case 3:** In this case, between jump times the process $\overline{\mathcal{X}}_t$ evolves as the process \mathcal{X}_t . The rate of the jumps is given by the function

$$\begin{aligned}\mathcal{W}_{t,\mu_t}(x) &:= \mu_t((\mathcal{V}_t - \mathcal{V}_t(x))_+) \\ &= \mu_t(\mathcal{V}_t \mathbf{1}_{\mathcal{V}_t \geq \mathcal{V}_t(x)}) - \mu_t(\mathcal{V}_t \geq \mathcal{V}_t(x)) \mathcal{V}_t(x).\end{aligned}$$

In other words, the jump times $(T_n)_{n \geq 0}$ are given by the following recursive formulae

$$T_{n+1} = \inf \left\{ t \geq T_n : \int_{T_n}^t \mathcal{W}_{s,\mu_t}(\overline{\mathcal{X}}_s) ds \geq e_n \right\}$$

where $T_0 = 0$, and $(e_n)_{n \geq 0}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n the process $\overline{\mathcal{X}}_{T_n-} = x$ jumps to new site $\overline{\mathcal{X}}_{T_n} = y$ randomly chosen with the distribution $\Psi_{(\mathcal{V}_{T_n} - \mathcal{V}_{T_n}(x))_+}(\mu_{T_n-})$.

For any $f \in D(L)$ we also have that

$$\Gamma_{\widehat{L}_{t,\mu_t}}(f, f)(x) = \widehat{L}_{t,\mu_t}((f - f(x))^2)(x) = \int [f(y) - f(x)]^2 (\mathcal{V}_t(y) - \mathcal{V}_t(x))_+ \mu_t(dy)$$

so that

$$\mu_t \left[\Gamma_{\widehat{L}_{t,\mu_t}}(f, f) \right] = \int [f(y) - f(x)]^2 (\mathcal{V}_t(y) - \mathcal{V}_t(x))_+ \mu_t(dx) \mu_t(dy). \quad (18)$$

We end this section with another McKean interpretation model combining cases 1 and 2, as an alternative to the generator described in the latter case. First, using the fact that

$$\mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_+ - [\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_-) = \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]) = 0$$

we prove the following decompositions

$$\begin{aligned}\mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t) \mu_t(f) &= \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)] f) \\ &= \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_+ f) - \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_- f) \\ &= \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_+ [f - \mu_t(f)]) \\ &\quad - \mu_t([\mathcal{V}_t - \mu_t(\mathcal{V}_t)]_- [f - \mu_t(f)]).\end{aligned}$$

Using the same line of arguments as those used in cases 1 and 2, this implies that

$$\mu_t(\mathcal{V}_t f) - \mu_t(\mathcal{V}_t) \mu_t(f) = \mu_t(\widehat{L}_{t,\mu_t}(f)) \quad \text{with} \quad \widehat{L}_{t,\mu_t} = \widehat{L}_{t,\mu_t}^+ + \widehat{L}_{t,\mu_t}^-$$

where the pair of interacting jump generators is given by

$$\widehat{L}_{t,\mu_t}^-(f)(x) = [\mathcal{V}_t(x) - \mu_t(\mathcal{V}_t)]_- \int [f(y) - f(x)] \mu_t(dy)$$

and

$$\widehat{L}_{t,\mu_t}^+(f)(x) = \int [f(y) - f(x)] [\mathcal{V}_t(y) - \mu_t(\mathcal{V}_t)]_+ \mu_t(dy).$$

In this situation, for any $f \in D(L)$ we also have that

$$\begin{aligned}\Gamma_{\widehat{L}_{t,\mu_t}}(f, f)(x) &= \Gamma_{\widehat{L}_{t,\mu_t}^+}(f, f)(x) + \Gamma_{\widehat{L}_{t,\mu_t}^-}(f, f)(x) \\ &= \int [f(y) - f(x)]^2 ([\mathcal{V}_t(y) - \mu_t(\mathcal{V}_t)]_+ + [\mathcal{V}_t(x) - \mu_t(\mathcal{V}_t)]_-) \mu_t(dy)\end{aligned}$$

so that

$$\mu_t \left[\Gamma_{\widehat{L}_{t,\mu_t}}(f, f) \right] = \int [f(y) - f(x)]^2 |\mathcal{V}_t(y) - \mu_t(\mathcal{V}_t)| \mu_t(dx) \mu_t(dy).$$

4.3 Mean field particle interpretation models

The mean field N -particle model $\xi_t := (\xi_t^i)_{1 \leq i \leq N}$ associated with a given collection of generators L_{t,μ_t} satisfying the weak equation (15) is a Markov process in E^N with infinitesimal generator given by the following formulae

$$\mathcal{L}_t(F)(x^1, \dots, x^N) := \sum_{1 \leq i \leq N} L_{t,m(x)}^{(i)}(F)(x^1, \dots, x^i, \dots, x^N) \quad \text{with} \quad m(x) := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{x^i} \quad (19)$$

for sufficiently regular functions F on E^N , and for any $x = (x^i)_{1 \leq i \leq N} \in E^N$. In the above formulae, $L_{t,m(x)}^{(i)}$ stands for the operator $L_{t,m(x)}$ acting on the function $x^i \mapsto F(x^1, \dots, x^i, \dots, x^N)$.

Before entering into the description of the particle model associated with the three cases presented in section 4.2, we provide a brief discussion of the convergence analysis of these stochastic models. Firstly, we recall that

$$dF(\xi_t) = \mathcal{L}_t(F)(\xi_t) dt + d\mathcal{M}_t(F)$$

for some martingale $\mathcal{M}_t(\varphi)$ with increasing process given by

$$\langle \mathcal{M}(F) \rangle_t := \int_0^t \Gamma_{\mathcal{L}_s}(F, F)(\xi_s) ds.$$

In the above we denote by $\Gamma_{\mathcal{L}_s}$ the carré du champ operator associated with \mathcal{L}_s , and defined by

$$\Gamma_{\mathcal{L}_s}(F, F)(x) := \mathcal{L}_s \left[(F - F(x))^2 \right] (x) = \mathcal{L}_s(F^2)(x) - F(x)\mathcal{L}_s(F)(x).$$

For empirical test functions of the following form $F(x) = m(x)(f)$, with $f \in D(L)$, we find that

$$\mathcal{L}_s(F)(x) = m(x)(L_{s,m(x)}(f)) \quad \text{and} \quad \Gamma_{\mathcal{L}_s}(\varphi, \varphi)(x) = \frac{1}{N} m(x) \left(\Gamma_{L_{s,m(x)}}(f, f) \right).$$

From this discussion, if we set $\mu_t^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i}$, then we find that

$$d\mu_t^N(f) = \mu_t^N(L_{t,\mu_t^N}(f)) dt + \frac{1}{\sqrt{N}} dM_t^N(f)$$

for any $f \in D(L)$, with the martingale $M_t^N(f) = \sqrt{N} \mathcal{M}_t(F)$ with angle bracket given by

$$\langle M^N(f) \rangle_t := \int_0^t \mu_s^N \left(\Gamma_{L_{s,\mu_s^N}}(f, f) \right) ds.$$

A more explicit description of the r.h.s. terms in the above can be given in the three cases discussed in section 4.2. For instance, in the third case, using formula (18) we find that

$$\langle M^N(f) \rangle_t := \int_0^t \int [f(y) - f(x)]^2 (\mathcal{V}_s(y) - \mathcal{V}_s(x))_+ \mu_s^N(dx) \mu_s^N(dy) ds.$$

We conclude that μ_t^N “almost solve”, as $N \uparrow \infty$, the nonlinear evolution equation (15). For a more thorough discussion of these continuous time models, we refer to the review article [15], and the references therein.

By construction, the generator \mathcal{L}_t associated with the nonlinear model (15) is decomposed into a mutation generator $\mathcal{L}_t^{\text{mut}}$ and an interacting jump generator $\mathcal{L}_t^{\text{jump}}$

$$\mathcal{L}_t = \mathcal{L}_t^{\text{mut}} + \mathcal{L}_t^{\text{jump}}$$

with $\mathcal{L}_t^{\text{mut}}$ and $\mathcal{L}_t^{\text{jump}}$ defined by

$$\begin{aligned} \mathcal{L}_t^{\text{mut}}(F)(x) &= \sum_{1 \leq i \leq N} L_t^{(i)}(F)(x^1, \dots, x^i, \dots, x^N) \\ \mathcal{L}_t^{\text{jump}}(F)(x) &= \sum_{1 \leq i \leq N} \widehat{L}_{t, m(x)}^{(i)}(F)(x^1, \dots, x^i, \dots, x^N). \end{aligned}$$

The mutation generator $\mathcal{L}_t^{\text{mut}}$ describes the evolution of the particles between the jumps. Between jumps, the particles evolve independently with L_t -motions in the sense that they explore the state space as independent copies of the process \mathcal{X}_t with generator L_t . The jump transition depends on the form of the generator \widehat{L}_{t, μ_t} .

- **Case 1:** In this situation the jump generator is given by

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \sum_{1 \leq i \leq N} \mathcal{U}_t(x^i) \int [F(\theta_u^i(x)) - F(x)] m(x)(du)$$

with the population mappings θ_u^i defined below

$$\theta_u^i : x \in E^N \mapsto \theta_u^i(x) = (x^1, \dots, x^{i-1}, \underbrace{u}_{i\text{-th}}, x^{i+1}, \dots, x^N) \in E^N.$$

The quantity $\mathcal{U}_t(x^i)$ represents the jump rate of the i -th particle ξ_t^i . More precisely, if we denote by T_n^i the n -th jump time of ξ_t^i , we have

$$T_{n+1}^i = \inf \left\{ t \geq T_n^i : \int_{T_n^i}^t \mathcal{U}_s(\xi_s^i) ds \geq e_n^i \right\} \quad (20)$$

where $(e_n^i)_{1 \leq i \leq N, n \in \mathbb{N}}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n^i the process $\xi_{T_n^i-}^i = x^i$ jumps to new site $\xi_{T_n^i}^i = u$ randomly chosen with the distribution $m(\xi_{T_n^i-}^i)(du)$. In other words, at the jump time the i -th particle jumps to a new state randomly chosen in the current population.

The probabilistic interpretation of the jump generator is not unique. For instance, it is easily checked that $\mathcal{L}_t^{\text{jump}}$ can be rewritten in the following form

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \lambda_t(x) \int [F(y) - F(x)] \mathcal{P}_t(x, dy)$$

with the population jump rate $\lambda_t(x)$ and the Markov transition $\mathcal{P}_t(x, dy)$ on E^N given below

$$\lambda_t(x) := Nm(x)(\mathcal{U}_t) \quad \text{and} \quad \mathcal{P}_t(x, dy) = \sum_{1 \leq i \leq N} \frac{\mathcal{U}_t(x^i)}{\sum_{1 \leq i' \leq N} \mathcal{U}_t(x^{i'})} \frac{1}{N} \sum_{1 \leq j \leq N} \delta_{\theta_{x_j}^i(x)}(dy).$$

In this interpretation, the individual jumps are replaced by population jumps at rate $\lambda_t(\xi_t)$. More precisely, the jump times T_n of the whole population are defined by

$$T_{n+1} = \inf \left\{ t \geq T_n : \int_{T_n}^t \left[\sum_{1 \leq i \leq N} \mathcal{U}_s(\xi_s^i) \right] ds \geq e_n \right\}$$

where $(e_n)_{n \in \mathbb{N}}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n the population $\xi_{T_n-} = x$ jumps to new population $\xi_{T_n} = y$ randomly chosen with the distribution $\mathcal{P}_{T_n-}(\xi_{T_n-}, dy)$. In other words, at the jump time T_n , we select randomly a state $\xi_{T_n-}^i$ with a probability proportional to $\mathcal{U}_t(\xi_{T_n-}^i)$, and we replace this state by a randomly chosen state $\xi_{T_n-}^j$ in the population, with $1 \leq j \leq N$. We end this description with an alternative interpretation when $\|\mathcal{U}_t\| \leq C$ for some finite constant $C < \infty$. In this situation, we clearly have $\|\lambda_t\| \leq NC$ and

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \lambda' \int [F(y) - F(x)] \mathcal{P}'_t(x, dy)$$

with the jump rate λ' and the Markov jump transitions \mathcal{P}'_t defined below

$$\lambda' = NC \quad \text{and} \quad \mathcal{P}'_t(x, dy) := \frac{\lambda_t(x)}{NC} \mathcal{P}_t(x, dy) + \left(1 - \frac{\lambda_t(x)}{NC}\right) \delta_x(dy).$$

In this interpretation, the population jump times T_n arrive at the higher rate $\lambda' = NC$. At the jump time T_n the population $\xi_{T_n-} = x$ jumps to new population $\xi_{T_n} = y$ randomly chosen with the distribution $\mathcal{P}'_{T_n-}(\xi_{T_n-}, dy)$.

In the models described above, as usual, between the jump times T_n of the population every particle evolves independently with L_t -motions.

- **Case 2 :** In this situation, the jump generator is given by

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \sum_{1 \leq i \leq N} m(x)(\mathcal{V}_t) \int [F(\theta_u^i(x)) - F(x)] \Psi_{\mathcal{V}_t}(m(x))(du).$$

The particles have a common jump rate given by the empirical average $m(\xi_t)(\mathcal{V}_t)$. In other words, the jump times T_n^i of a particle ξ_t^i are given by the following recursive formulae

$$T_{n+1}^i = \inf \left\{ t \geq T_n^i : \int_{T_n^i}^t m(\xi_s)(\mathcal{V}_s) ds \geq e_n^i \right\}$$

where $(e_n^i)_{1 \leq i \leq N, n \geq 0}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n^i the process $\xi_{T_n^i-}^i = x^i$ jumps to new site $\xi_{T_n^i}^i = u$ randomly chosen with the weighted distribution $\Psi_{\mathcal{V}_{T_n^i-}}(m(\xi_{T_n^i-}^i))(du)$.

As mentioned in the first case, the probabilistic interpretation of the jump generator is not unique. In this situation, it is easily checked that $\mathcal{L}_t^{\text{jump}}$ can be rewritten in the following form

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \lambda_t(x) \int [F(y) - F(x)] \mathcal{P}_t(x, dy)$$

with the population jump rate λ_t and the Markov transition $\mathcal{P}_t(x, dy)$ on E^N defined below

$$\lambda_t(x) := Nm(x)(\mathcal{V}_t) \quad \text{and} \quad \mathcal{P}_t(x, dy) = \frac{1}{N} \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq N} \frac{\mathcal{V}_t(x^j)}{\sum_{1 \leq j' \leq N} \mathcal{V}_t(x^{j'})} \delta_{\theta_{x^j}^i(x)}(dy).$$

The description of the evolution of the population model follows the same lines as the ones given in case 1.

- **Case 3 :** In this situation, the jump generator is given by

$$\begin{aligned} \mathcal{L}_t^{\text{jump}}(F)(x) &= \sum_{1 \leq i \leq N} \int [F(\theta_u^i(x)) - F(x)] (\mathcal{V}_t(u) - \mathcal{V}_t(x^i))_+ m(x)(du) \\ &= \sum_{1 \leq i \leq N} m(x)((\mathcal{V}_t - \mathcal{V}_t(x^i))_+) \int [F(\theta_u^i(x)) - F(x)] \Psi_{(\mathcal{V}_t - \mathcal{V}_t(x^i))_+}(m(x))(du). \end{aligned}$$

In this interpretation, the jump rate of the i -th particle is given by the average potential variation of the particle with higher values

$$m(x)((\mathcal{V}_t - \mathcal{V}_t(x^i))_+) = \frac{1}{N} \sum_{1 \leq j \leq N} 1_{\{\mathcal{V}_t(x^j) > \mathcal{V}_t(x^i)\}} (\mathcal{V}_t(x^j) - \mathcal{V}_t(x^i)).$$

More precisely, if we denote by T_n^i the n -th jump time of ξ_t^i , we have

$$T_{n+1}^i = \inf \left\{ t \geq T_n^i : \int_{T_n^i}^t m(\xi_s)((\mathcal{V}_s - \mathcal{V}_s(\xi_s^i))_+) ds \geq e_n^i \right\}$$

where $(e_n^i)_{1 \leq i \leq N, n \in \mathbb{N}}$ stands for a sequence of i.i.d. exponential random variables with unit parameter. At the jump time T_n^i the particle $\xi_{T_n^i-}^i = x^i$ jumps to new site $\xi_{T_n^i}^i = u$ randomly chosen with the distribution

$$\Psi_{(\mathcal{V}_{T_n^i-} - \mathcal{V}_{T_n^i-}(x^i))_+}(m(\xi_{T_n^i-}^i))(du) \propto \sum_{1 \leq j \leq N} 1_{\{\mathcal{V}_{T_n^i-}(\xi_{T_n^i-}^j) > \mathcal{V}_{T_n^i-}(x^i)\}} (\mathcal{V}_{T_n^i-}(\xi_{T_n^i-}^j) - \mathcal{V}_{T_n^i-}(x^i)) \delta_{\xi_{T_n^i-}^j}(du).$$

In other words, we choose randomly a new site $\xi_{T_n^i}^i = \xi_{T_n^i-}^j$, among the ones with higher potential value with a probability proportional to the difference of potential $(\mathcal{V}_{T_n^i-}(\xi_{T_n^i-}^j) - \mathcal{V}_{T_n^i-}(\xi_{T_n^i-}^i))$.

As the first two cases discussed above, we can also interpret this jump generator at the level of the population. In this interpretation we have

$$\mathcal{L}_t^{\text{jump}}(F)(x) = \lambda_t(x) \int [F(y) - F(x)] \mathcal{P}_t(x, dy)$$

with the population jump rate

$$\lambda_t(x) = N \int m(x)(du) \int m(x)(dv) (\mathcal{V}_t(u) - \mathcal{V}_t(v))_+$$

and the population jump transition

$$\mathcal{P}_t(x, dy) = \sum_{1 \leq i, j \leq N} \frac{(\mathcal{V}_t(x^j) - \mathcal{V}_t(x^i))_+}{\sum_{1 \leq i', j' \leq N} (\mathcal{V}_t(x^{j'}) - \mathcal{V}_t(x^{i'}))_+} \delta_{\theta_{x^j}^i}(dy) .$$

Remark 4.1: In case **(D)**, the reference Markov process $\mathcal{X}_t = X_{[t]}$ has deterministic and fixed time jumps on integer times so that the generator approach developed above does not apply directly. Nevertheless their probabilistic interpretation is defined in the same way:

Between the jumps, the process \overline{X}_t evolves as \mathcal{X}_t , and the N particles explore the state space as independent copies of the process \mathcal{X}_t . The rate of the jumps and their random spatial location are defined using the same interpretations as the ones given above.

The stochastic modeling and the analysis of these continuous time models and their particle interpretations can be developed using the semigroup techniques provided in [25].

5 Discrete time models

5.1 McKean models and Feynman-Kac semigroups

As in the continuous time case, these discrete time evolution equations (7) can be interpreted as the evolution of the laws defined by $\text{Law}(\overline{X}_{t_n}) = \mu_{t_n}^{(m)}$ of a time inhomogeneous Markov process \overline{X}_{t_n} with Markov transitions generators $\mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^{(m)}}$ that depend on the distribution of the random states at the previous sub-integer mesh time increment. This probabilistic model is also called the McKean interpretation of the evolution equation (7) in terms of a time inhomogeneous Markov chain. By construction, the elementary transitions of the Markov chain $\overline{X}_{t_n} \rightsquigarrow \overline{X}_{t_{n+1}}$ are decomposed into two separate transitions $\overline{X}_{t_n} \rightsquigarrow \widehat{X}_{t_n} \rightsquigarrow \overline{X}_{t_{n+1}}$.

First, the state $\overline{X}_{t_n} = x$ jumps to a new location $\widehat{X}_{t_n} = y$ randomly chosen with the Markov transition $\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(x, dy)$, given by one of the three cases presented in section 2.2 when applied to the particle approximation of the measure $\mu_{t_n}^{(m)}$ at mesh time increment t_n . Then, the selected state $\widehat{X}_{t_n} = y$ evolves to a new site $\overline{X}_{t_{n+1}} = z$ according to the Markov transition $\mathcal{M}_{t_n, t_{n+1}}(y, dz)$.

Next, we recall some basic properties of the semigroup $\Phi_{t_p, t_n}^{(m)}$ of the flow of measures $\mu_{t_n}^{(m)}$. By construction, we have

$$\Phi_{t_p, t_n}^{(m)}(\mu_{t_p}^{(m)})(f) = \mu_{t_p}^{(m)} Q_{t_p, t_n}^{(m)}(f) / \mu_{t_p}^{(m)} Q_{t_p, t_n}^{(m)}(1)$$

with Feynman-Kac semigroup $Q_{t_p, t_n}^{(m)}$ defined by

$$Q_{t_p, t_n}^{(m)}(f)(x) = \mathbb{E} \left(f(\mathcal{X}_{t_n}) \prod_{p \leq q < n} e^{\mathcal{V}_{t_q}(\mathcal{X}_{t_q})/m} \mid \mathcal{X}_{t_p} = x \right). \quad (21)$$

We notice that when we consider the Boltzmann-Gibbs transformation associated with the potential function $G_{t_p, t_n}^{(m)} = Q_{t_p, t_n}^{(m)}(1)$ then the semigroup of the flow of measures $\mu_{t_n}^{(m)}$ can be expressed according to

$$\Phi_{t_p, t_n}^{(m)}(\mu_{t_p}^{(m)}) = \Psi_{G_{t_p, t_n}^{(m)}}(\mu_{t_p}^{(m)}) P_{t_p, t_n}^{(m)} \quad \text{with} \quad P_{t_p, t_n}^{(m)}(f) = Q_{t_p, t_n}^{(m)}(f)/Q_{t_p, t_n}^{(m)}(1). \quad (22)$$

Definition 5.1 *We consider the integral operators*

$$L_{t_n, \mu}^{(m)} := \mathcal{K}_{t_n, t_{n+1}, \mu} - Id, \quad L_{t_n}^{(m)} := \mathcal{M}_{t_n, t_{n+1}} - Id \quad \text{and} \quad \widehat{L}_{t_n, \mu}^{(m)} := \mathcal{S}_{t_n, \mu} - Id.$$

Lemma 5.2 *We have the decomposition*

$$L_{t_n, \mu}^{(m)} = L_{t_n}^{(m)} + \widehat{L}_{t_n, \mu}^{(m)} + \widehat{L}_{t_n, \mu}^{(m)} L_{t_n}^{(m)}.$$

In addition, for any $f \in \mathcal{B}_b(E)$ $\mu \in \mathcal{P}(E)$, and any $x \in E$, we have

$$\mathcal{K}_{t_n, t_{n+1}, \mu} \left([f - \mathcal{K}_{t_n, t_{n+1}, \mu}(f)(x)]^2 \right)(x) = \Gamma_{L_{t_n, \mu}^{(m)}}(f, f)(x) - \left(L_{t_n, \mu}^{(m)}(f)(x) \right)^2.$$

Proof: Using the decomposition

$$L_{t_n, \mu}^{(m)} = (\mathcal{M}_{t_n, t_{n+1}} - Id) + (\mathcal{S}_{t_n, \mu} - Id) + (\mathcal{S}_{t_n, \mu} - Id)(\mathcal{M}_{t_n, t_{n+1}} - Id)$$

we readily check the first assertion. We prove the second decomposition using the fact that

$$[\mathcal{K}_{t_n, t_{n+1}, \mu}(f)]^2 = \left(L_{t_n, \mu}^{(m)}(f) \right)^2 + f^2 - 2f L_{t_n, \mu}^{(m)}(f).$$

This ends the proof of the lemma. ■

In the further development of this section $c_{t_n} < \infty$ stands for some generic finite constant whose values may vary from line to line. For any function $f \in D(L)$, such that $L_t(f) \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$, we also define

$$\|f\|_{t_n} := \|f\| + \sup_{t_n \leq t \leq t_{n+1}} (\|\partial L_t(f)/\partial t\| + \|L_t(f)\| + \|L_t^2(f)\|). \quad (23)$$

Proposition 5.3 *In case (D), we have*

$$L_{t_n}^{(m)} = 1_{\mathbb{N}}(t_{n+1}) (M_{t_{n+1}} - Id) \quad \text{and} \quad L_{t_n, \mu}^{(m)} = \widehat{L}_{t_n, \mu}^{(m)} + 1_{\mathbb{N}}(t_{n+1}) \mathcal{S}_{t_n, \mu} (M_{t_{n+1}} - Id).$$

In case (C), we have the first order expansion

$$L_{t_n}^{(m)}(f) = L_{t_n}(f) \frac{1}{m} + R_{t_n}(f) \frac{1}{m^2}. \quad (24)$$

for any function $f \in D(L)$, such that $L_t(f) \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$, with some remainder operator R_{t_n} such that $\|R_{t_n}(f)\| \leq c_{t_n} \|f\|_{t_n}$. Furthermore, we have the first order expansion

$$L_{t_n, \mu}^{(m)}(f) = \frac{1}{m} L_{t_n, \mu}(f) + \frac{1}{m^2} R_{t_n, \mu}(f) \quad (25)$$

with some second order remainder term $R_{t_n, \mu}(f)$ such that $\sup_{\mu \in \mathcal{P}(E)} \|R_{t_n, \mu}(f)\| \leq c_{t_n} \|f\|_{t_n}$. In addition, we have

$$\mathcal{K}_{t_n, t_{n+1}, \mu} \left([f - \mathcal{K}_{t_n, t_{n+1}, \mu}(f)(x)]^2 \right) (x) = \Gamma_{L_{t_n, \mu}}(f, f)(x) \frac{1}{m} + \mathcal{R}_{t_n, \mu}(f, f)(x) \frac{1}{m^2} \quad (26)$$

with some remainder operator s.t. $\sup_{\mu \in \mathcal{P}(E)} \|\mathcal{R}_{t_n, \mu}(f, f)\| \leq c_{t_n} \|f^2\|_{t_n}$.

The proof of proposition 5.3 is provided in the appendix, on page 36.

The first order expansions stated in the proposition 5.3 can be used to develop a stochastic perturbation approach to estimate the deviations of the measures $\mu_{t_n}^{(m)}$ around their limiting values μ_{t_n} . Next, we provide an alternative approach based on the explicit representation (17) of the time inhomogeneous transition of the limiting process \mathcal{X}_{t_n} on the time mesh sequence t_n . In the first case discussed on page 10, we have

$$\mu_{t_{n+1}}^{(m)} = \Phi_{t_n, t_{n+1}}^{(m)} \left(\mu_{t_n}^{(m)} \right) := \Psi_{e^{-\mathcal{U}_{t_n}/m}} \left(\mu_{t_n}^{(m)} \right) \mathcal{M}_{t_n, t_{n+1}} = \mu_{t_n}^{(m)} \overline{\mathcal{P}}_{t_n, t_{n+1}, \mu_{t_n}^{(m)}}^{(m)} \quad (27)$$

with the Markov transition

$$\begin{aligned} & \overline{\mathcal{P}}_{t_n, t_{n+1}, \mu_{t_n}^{(m)}}^{(m)}(x, dy) \\ &= e^{-\mathcal{U}_{t_n}(x)/m} \mathcal{M}_{t_n, t_{n+1}}(x, dy) + (1 - e^{-\mathcal{U}_{t_n}(x)/m}) \Psi_{e^{-\mathcal{U}_{t_n}/m}} \left(\mu_{t_n}^{(m)} \right) \mathcal{M}_{t_n, t_{n+1}}(dy). \end{aligned} \quad (28)$$

Using (17) we readily find that

$$\mu_{t_{n+1}}^{(m)} = \mu_{t_n}^{(m)} \overline{\mathcal{P}}_{t_n, t_{n+1}, \mu_{t_n}^{(m)}}^{(m)} + \frac{1}{m} \mathcal{W}_{t_n, t_{n+1}}^{(m)} = \Phi_{t_n, t_{n+1}} \left(\mu_{t_n}^{(m)} \right) + \frac{1}{m} \mathcal{W}_{t_n, t_{n+1}}^{(m)}$$

with the signed measure

$$\mathcal{W}_{t_n, t_{n+1}}^{(m)} := \mu_{t_n}^{(m)} \mathcal{R}_{t_n, t_{n+1}, \mu_{t_n}^{(m)}}^{(m)} \quad \text{s.t.} \quad \sup_{m \geq 1} \left\| \mathcal{W}_{t_n, t_{n+1}}^{(m)} \right\|_{\text{tv}} \leq c_{t_n}$$

for some finite constant whose values only depend on the potential function \mathcal{U}_t . In summary, we have proven the following first order local perturbation decompositions

$$\begin{aligned} \mu_{t_{n+1}}^{(m)} &= \Phi_{t_n, t_{n+1}} \left(\mu_{t_n}^{(m)} \right) + \frac{1}{m} \mathcal{W}_{t_n, t_{n+1}}^{(m)} \\ \mu_{t_{n+1}} &= \Phi_{t_n, t_{n+1}}(\mu_{t_n}) \end{aligned}$$

These local expansions allow the use of perturbation theory developed in section 7.1 of [10] to derive several qualitative estimates between $\mu_{t_n}^{(m)}$ and μ_{t_n} in terms of the stability properties of the Feynman-Kac semigroup $\Phi_{t_n, t_{n+1}}$.

5.2 Mean field particle interpretation models

If we set $\mu_{t_n}^N = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_{t_n}^i}$, then we have the decomposition

$$\mu_{t_{n+1}}^N = \mu_{t_n}^N \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N} + \frac{1}{\sqrt{N}} W_{t_n, t_{n+1}}^N$$

with the sequence of empirical random fields $W_{t_n, t_{n+1}}^N$ such that

$$\mathbb{E} \left(W_{t_n, t_{n+1}}^N(f) \mid \xi_{t_n} \right) = 0$$

and

$$\mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^2 \mid \xi_{t_n} \right) = \int \mu_{t_n}^N(du) \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u, dv) \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^2.$$

As for the continuous time models, we conclude that $\mu_{t_n}^N$ “almost solve”, as $N \uparrow \infty$, the nonlinear evolution equation (7). For a more thorough discussion of these local sampling random field models, we refer the reader to [10, 15, 16], and references therein.

By construction, the elementary transitions of the Markov chain $\xi_{t_n} \rightsquigarrow \xi_{t_{n+1}}$ are decomposed into two separate transitions:

$$\xi_{t_n} \rightsquigarrow \widehat{\xi}_{t_n} = \left(\widehat{\xi}_{t_n}^i \right)_{1 \leq i \leq N} \rightsquigarrow \xi_{t_{n+1}}. \quad (29)$$

First, every particle $\xi_{t_n}^i = x^i$ jumps independently to a new location $\widehat{\xi}_{t_n}^i = y^i$ randomly chosen with the Markov transition $\mathcal{S}_{t_n, m(\xi_{t_n})}(x^i, dy^i)$, with $1 \leq i \leq N$. Following this, each particle $\widehat{\xi}_{t_n}^i = y^i$ evolves independently to a new site $\xi_{t_{n+1}}^i = z^i$ according to the Markov transition $\mathcal{M}_{t_n, t_{n+1}}(y^i, dz^i)$, with $1 \leq i \leq N$.

In other words, the mutation transition describes the evolution of the particles between the jumps. Between the jumps, the particles evolve independently with $\mathcal{M}_{t_n, t_{n+1}}$ -motions in the sense that they explore the state space as independent copies of the process \mathcal{X}_{t_n} with Markov transition $\mathcal{M}_{t_n, t_{n+1}}$. The jump transition can also be interpreted as an acceptance-rejection transition equipped with a recycling mechanism. In this interpretation, the mutation transition can be interpreted as a proposal transition. Notice that the selection type transition is dictated by the choice of the transition $\mathcal{S}_{t_n, \mu_{t_n}^N}$.

We illustrate these jump type transitions in the first case presented on page 5. In this situation, we recall that the selection transition of the i -th particle $\xi_{t_n}^i \rightsquigarrow \widehat{\xi}_{t_n}^i$ is given by the following distribution

$$\mathcal{S}_{t_n, \mu_{t_n}^N}(\xi_{t_n}^i, dy) := e^{-\mathcal{U}_{t_n}(\xi_{t_n}^i)/m} \delta_{\xi_{t_n}^i}(dy) + \left(1 - e^{-\mathcal{U}_{t_n}(\xi_{t_n}^i)/m} \right) \Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^N)}(dy). \quad (30)$$

Next, we provide an interpretation of this transition as an acceptance-rejection scheme with a recycling mechanism. We let $\widetilde{\xi}_{t_n} = \left(\widetilde{\xi}_{t_n}^i \right)_{1 \leq i \leq N}$ be a sequence of conditionally independent random variables with common law

$$\Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^N)} = \sum_{1 \leq i \leq N} \frac{e^{-\mathcal{U}_{t_n}(\xi_{t_n}^i)/m}}{\sum_{1 \leq j \leq N} e^{-\mathcal{U}_{t_n}(\xi_{t_n}^j)/m}} \delta_{\xi_{t_n}^i}.$$

We also consider a sequence of conditionally independent Bernoulli random variables with distribution

$$\mathbb{P}(\epsilon_{t_n}^i = 1 \mid \xi_{t_n}) = 1 - \mathbb{P}(\epsilon_{t_n}^i = 0 \mid \xi_{t_n}) = e^{-\mathcal{U}_{t_n}(\xi_{t_n}^i)/m}.$$

In this notation, we have that

$$\widehat{\xi}_{t_n}^i = \epsilon_{t_n}^i \xi_{t_n}^i + (1 - \epsilon_{t_n}^i) \widetilde{\xi}_{t_n}^i.$$

In other words, the particle $\xi_{t_n}^i$ is accepted when $\epsilon_{t_n}^i = 1$; otherwise, it is rejected and replaced by a particle $\widetilde{\xi}_{t_n}^i$ randomly chosen with the updated weighted distribution $\Psi_{e^{-\mathcal{U}_{t_n}/m}}(\mu_{t_n}^N)$. The pool of particles that have been accepted from the start provide a sequence of exact samples. More precisely, it can be easily shown that

$$\text{Law}(\xi_{t_n}^i \mid \forall p : 0 \leq p < n, \epsilon_{t_p}^i = 1) = \mu_{t_n}.$$

See for instance section 1.5.1 in [10].

In connection with (20), we notice that the rejection times T_n^i on the time mesh $(t_q)_{q \geq 0}$ can be defined as follows

$$\begin{aligned} T_{n+1}^i &= \inf \left\{ t_p > T_n^i : \sum_{T_n^i \leq k \leq t_p} \mathcal{U}_{t_k}(\xi_{t_k}^i)/m \geq e_n^i \right\} \\ &= \inf \left\{ t_p > T_n^i : \prod_{T_n^i \leq k \leq t_p} e^{-\mathcal{U}_{t_k}(\xi_{t_k}^i)/m} \leq u_n^i \right\} \end{aligned}$$

where $(u_n^i)_{1 \leq i \leq N, n \in \mathbb{N}}$ stands for a sequence of i.i.d. uniform random variables on $]0, 1]$, and $e_n^i := -\log u_n^i$, with $1 \leq i \leq N, n \in \mathbb{N}$, is the corresponding sequence of i.i.d. exponential random variables with unit parameter. We check this claim using the following observations

$$\begin{aligned} &\mathbb{P}(T_{n+1}^i = t_p \mid T_n^i, \forall T_n^i \leq k \leq t_p \xi_{t_k}^i) \\ &= \mathbb{P}(\forall T_n^i \leq k < t_p \epsilon_{t_k}^i = 1, \epsilon_{t_p}^i = 0 \mid T_n^i, \forall T_n^i \leq k \leq t_p \xi_{t_k}^i) \\ &= \left(\prod_{T_n^i \leq k < t_p} e^{-\mathcal{U}_{t_k}(\xi_{t_k}^i)/m} \right) \times \left(1 - e^{-\mathcal{U}_{t_p}(\xi_{t_p}^i)/m} \right) \\ &= e^{-\sum_{T_n^i \leq k < t_p} \mathcal{U}_{t_k}(\xi_{t_k}^i)/m} \times \left(1 - e^{-\mathcal{U}_{t_p}(\xi_{t_p}^i)/m} \right) \\ &= \mathbb{P}\left(\sum_{T_n^i \leq k < t_p} \mathcal{U}_{t_k}(\xi_{t_k}^i)/m < e_n^i \leq \sum_{T_n^i \leq k \leq t_p} \mathcal{U}_{t_k}(\xi_{t_k}^i)/m \mid T_n^i, \forall T_n^i \leq k \leq t_p \xi_{t_k}^i \right). \end{aligned}$$

At the jump time T_n^i the process $\xi_{T_n^i}^i = x^i$ jumps to new site $\widehat{\xi}_{T_n^i}^i = u$ randomly chosen with the distribution

$$\Psi_{e^{-\mathcal{U}_{T_n^i}/m}}(\mu_{T_n^i}^N). \quad (31)$$

5.3 An mean field model with uniform recycling

When m is large enough, the recycling distribution (31) in the Markov transition (30) is almost equal to $\mu_{T_n^i}^N$. For instance, we have the total variation estimate

$$\left\| \Psi_{e^{-\mathcal{U}_{T_n^i}/m}}(\mu_{T_n^i}^N) - \mu_{T_n^i}^N \right\|_{\text{tv}} \leq \left\| 1 - e^{-\mathcal{U}_{T_n^i}/m} \right\| \leq \|\mathcal{U}_{T_n^i}\|/m.$$

We prove these inequalities using the decomposition

$$\Psi_{e^{-\mathcal{U}/m}}(\mu)(f) - \mu(f) = \mu \left(\left[1 - e^{-\mathcal{U}/m} \right] [\Psi_{e^{-\mathcal{U}/m}}(\mu)(f) - f] \right)$$

which is valid for any bounded functions \mathcal{U} and f . Hence, to save computational time we can replace the recycling weighted measure $\Psi_{e^{-\mathcal{U}_{T_n^i}/m}}(\mu_{T_n^i}^N)$ by $\mu_{T_n^i}^N$. In this situation, the selection transition takes the following form

$$\tilde{\xi}_{t_n}^i = \epsilon_{t_n}^i \xi_{t_n}^i + (1 - \epsilon_{t_n}^i) \xi_{t_n}^{1 + \lfloor N\tau_{t_n}^i \rfloor}$$

where $\tau_{t_n}^i$ stands for a sequence of i.i.d. uniform random variables on $]0, 1]$.

Next, we detail some analysis of the differences between the McKean models with recycling Boltzmann-Gibbs transitions, and the models discussed which utilise uniform recycling.

We let $\tilde{\mathcal{S}}_{t_n, \mu}$ the collection of selection transitions defined as in (30), by replacing $\Psi_{e^{-\mathcal{U}_{t_n}/m}}(\mu)$ by the measure μ . Furthermore, we denote by $\Phi_{t_p, t_n}^{(m)}$, resp. $\tilde{\Phi}_{t_p, t_n}^{(m)}$, with $p \leq n$, the semigroups associated with these flows

$$\tilde{\Phi}_{t_p, t_n}^{(m)} \left(\tilde{\mu}_{t_p}^{(m)} \right) = \tilde{\mu}_{t_n}^{(m)} \quad \text{and} \quad \Phi_{t_p, t_n}^{(m)} \left(\mu_{t_p}^{(m)} \right) = \mu_{t_n}^{(m)}.$$

Then, using the decomposition

$$\left[\mathcal{S}_{t_n, \mu} - \tilde{\mathcal{S}}_{t_n, \mu} \right] (x, dy) = \left(1 - e^{-\mathcal{U}_{t_n}(x)/m} \right) (\Psi_{e^{-\mathcal{U}/m}}(\mu) - \mu) (dy)$$

we find that

$$\sup_{x \in E} \left\| \mathcal{S}_{t_n, \mu}(x, \cdot) - \tilde{\mathcal{S}}_{t_n, \mu}(x, \cdot) \right\|_{\text{tv}} \leq \|\mathcal{U}_{t_n}\|^2 / m^2.$$

Replacing $\Psi_{e^{-\mathcal{U}_{t_n}/m}}(\mu)$ by the measure μ in (28), the evolution equation (27) takes the following form

$$\tilde{\mu}_{t_{n+1}}^{(m)} = \tilde{\Phi}_{t_n, t_{n+1}}^{(m)} \left(\tilde{\mu}_{t_n}^{(m)} \right) := \tilde{\mu}_{t_n}^{(m)} \tilde{\mathcal{P}}_{t_n, t_{n+1}, \tilde{\mu}_{t_n}^{(m)}}^{(m)} \quad \text{with} \quad \tilde{\mathcal{P}}_{t_n, t_{n+1}, \tilde{\mu}_{t_n}^{(m)}}^{(m)} = \tilde{\mathcal{S}}_{t_n, \tilde{\mu}_{t_n}^{(m)}} \mathcal{M}_{t_n, t_{n+1}}.$$

From previous estimates, we find that

$$\tilde{\Phi}_{t_n, t_{n+1}}^{(m)} (\mu) = \Phi_{t_n, t_{n+1}}^{(m)} (\mu) + \frac{1}{m^2} \mathcal{R}_{t_n, t_{n+1}}^{(m)} (\mu)$$

with some measures $\mathcal{R}_{t_n, t_{n+1}}^{(m)} (\mu)$ s.t.

$$\sup_{\mu \in \mathcal{P}(E)} \left\| \mathcal{R}_{t_n, t_{n+1}}^{(m)} (\mu) \right\|_{\text{tv}} \leq \|\mathcal{U}_{t_n}\|^2.$$

We end this section with an estimate of the difference between the flow of measures $\tilde{\mu}_{t_{n+1}}^{(m)}$, and $\mu_{t_{n+1}}^{(m)}$. We are now in position to state, and to prove the following theorem.

Theorem 5.4 *For any $n \geq 0$, we have*

$$\left\| \tilde{\mu}_{t_n}^{(m)} - \mu_{t_n}^{(m)} \right\|_{\text{tv}} \leq c_{t_n} / m$$

for some finite constant $c_{t_n} < \infty$, whose values don't depend on the parameter m .

Proof: We use the stochastic perturbation analysis developed in section 6.3 in [16] (see also chapter 7 in [10]). We denote by $\Phi_{t_p, t_n}^{(m)}$, resp. $\tilde{\Phi}_{t_p, t_n}^{(m)}$, with $p \leq n$, the semigroups associated with these flows

$$\tilde{\Phi}_{t_p, t_n}^{(m)}(\tilde{\mu}_{t_p}^{(m)}) = \tilde{\mu}_{t_n}^{(m)} \quad \text{and} \quad \Phi_{t_p, t_n}^{(m)}(\mu_{t_p}^{(m)}) = \mu_{t_n}^{(m)}$$

Using the interpolating sequence of measures

$$0 \leq p \leq n \mapsto \Phi_{t_p, t_n}^{(m)}(\tilde{\Phi}_{t_0, t_p}^{(m)}(\tilde{\mu}_{t_0}^{(m)})) = \Phi_{t_p, t_n}^{(m)}(\tilde{\mu}_{t_p}^{(m)})$$

from the distribution

$$\Phi_{t_0, t_n}^{(m)}(\tilde{\Phi}_{t_0, t_0}^{(m)}(\tilde{\mu}_{t_0}^{(m)})) = \Phi_{t_0, t_n}^{(m)}(\tilde{\mu}_{t_0}^{(m)}) = \mu_{t_n}^{(m)} \quad \text{to the measure} \quad \Phi_{t_n, t_n}^{(m)}(\tilde{\Phi}_{t_0, t_n}^{(m)}(\tilde{\mu}_{t_0}^{(m)})) = \tilde{\mu}_{t_n}^{(m)}$$

Recalling that $\tilde{\mu}_{t_0}^{(m)} = \mu_{t_0}^{(m)}$, we find that

$$\begin{aligned} \tilde{\mu}_{t_n}^{(m)} - \mu_{t_n}^{(m)} &= \sum_{q=1}^n \left[\Phi_{t_q, t_n}^{(m)}(\tilde{\Phi}_{t_0, t_q}^{(m)}(\tilde{\mu}_{t_0}^{(m)})) - \Phi_{t_{q-1}, t_n}^{(m)}(\tilde{\Phi}_{t_0, t_{q-1}}^{(m)}(\tilde{\mu}_{t_0}^{(m)})) \right] \\ &= \sum_{q=1}^n \left[\Phi_{t_q, t_n}^{(m)}\left(\Phi_{t_{q-1}, t_q}^{(m)}(\tilde{\mu}_{t_{q-1}}^{(m)}) + \frac{1}{m^2} \mathcal{R}_{t_{q-1}, t_q}^{(m)}(\tilde{\mu}_{t_{q-1}}^{(m)})\right) - \Phi_{t_q, t_n}^{(m)}(\Phi_{t_{q-1}, t_q}^{(m)}(\tilde{\mu}_{t_{q-1}}^{(m)}) \right] \end{aligned}$$

By (22), we prove that

$$\begin{aligned} \left[\Phi_{t_p, t_n}^{(m)}(\mu) - \Phi_{t_p, t_n}^{(m)}(\nu) \right](f) &= \left[\Psi_{G_{t_p, t_n}^{(m)}}(\mu) - \Psi_{G_{t_p, t_n}^{(m)}}(\nu) \right] P_{t_p, t_n}^{(m)}(f) \\ &= \frac{\nu(G_{t_p, t_n}^{(m)})}{\mu(G_{t_p, t_n}^{(m)})} [\mu - \nu] \left(\frac{G_{t_p, t_n}^{(m)}}{\nu(G_{t_p, t_n}^{(m)})} \left[P_{t_p, t_n}^{(m)}(f) - \Psi_{G_{t_p, t_n}^{(m)}}(\nu) P_{t_p, t_n}^{(m)}(f) \right] \right) \end{aligned}$$

This implies that

$$\left\| \Phi_{t_p, t_n}^{(m)}(\mu) - \Phi_{t_p, t_n}^{(m)}(\nu) \right\|_{\text{tv}} \leq 2 g_{t_p, t_n}^{(m)} \beta(P_{t_p, t_n}^{(m)}) \|\mu - \nu\|_{\text{tv}}$$

with the Dobrushin contraction coefficient $\beta(P_{t_p, t_n}^{(m)}) (\leq 1)$, and the parameters

$$g_{t_p, t_n}^{(m)} := \sup_{x, y} \frac{G_{t_p, t_n}^{(m)}(x)}{G_{t_p, t_n}^{(m)}(y)} \leq \exp(2t_n \sup_{t \in [0, t_n]} \|\mathcal{U}_t\|).$$

This yields the rather crude estimates

$$\begin{aligned} m \left\| \tilde{\mu}_{t_n}^{(m)} - \mu_{t_n}^{(m)} \right\|_{\text{tv}} &\leq 2m^{-1} \sum_{p=1}^n g_{t_p, t_n}^{(m)} \beta(P_{t_p, t_n}^{(m)}) \|\mathcal{U}_{t_{q-1}}\|^2 \\ &\leq 2t_n \exp(2t_n \sup_{t \in [0, t_n]} \|\mathcal{U}_t\|) \sup_{t \in [0, t_n]} \|\mathcal{U}_t\|^2 \end{aligned}$$

This ends the proof of the theorem. ■

Working a little harder, under some regularity conditions, the estimates developed in the proof of the theorem can be used to obtain uniform estimates w.r.t. the time parameter. For instance, in case **(D)**, under the stability conditions (10), the constant c_{t_n} in theorem 5.4 can be chosen so that $\sup_n c_{t_n} < \infty$. We can extend these uniform results to continuous time models, using the stability analysis of continuous Feynman-Kac semigroups developed in [23].

6 First order decompositions

The main objective of this section is to prove theorem 3.2. In the further development of this section we let c , c_n , c_{t_n} , and $c_{t_n}(f)$ be, respectively, some universal constant, and some finite constants that depend on the parameters n , t_n , and the pair (t_n, f) , with values that may vary from line to line but do not depend on the parameters m and N . We also assume that m is chosen so that $\|\mathcal{V}_{t_n}\| \leq c_{t_n} m$, for any $n \geq 0$, and $N \geq m$.

6.1 Continuous time models

We start with the continuous time case **(C)** presented on page 4. By lemma 5.2 we find that

$$\mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^2 \mid \xi_{t_n} \right) = \mu_{t_n}^N \left[\Gamma_{L_{t_n, \mu_{t_n}^N}^{(m)}}(f, f) \right] - \mu_{t_n}^N \left(\left(L_{t_n, \mu_{t_n}^N}^{(m)}(f) \right)^2 \right). \quad (32)$$

Using (26), for any function $f \in D(L)$, such that $L_t(f), L_t(f^2) \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$, we find that

$$\mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^2 \mid \xi_{t_n} \right) = \mu_{t_n}^N \left[\Gamma_{L_{t_n, \mu_{t_n}^N}^{(m)}}(f, f) \right] \frac{1}{m} + R_{t_n}^N(f) \frac{1}{m^2}$$

with some remainder term $R_{t_n}^N(f)$ such that

$$\sup_{N \geq 1} |R_{t_n}^N(f)| \leq c_{t_n} \|f^2\|_{t_n}$$

with the norm $\|f\|_{t_n}$ of a function $f \in D(L)$, such that $L_t(f) \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$ defined in (23) By (21), we have

$$Q_{t_p, t_n}^{(m)}(f)(x) = \mathbb{E} \left(f(\mathcal{X}_{t_n}) \exp \left[\int_{t_p}^{t_n} \mathcal{V}_{\mathcal{I}(s)}(\mathcal{X}_{\mathcal{I}(s)}) ds \right] \mid \mathcal{X}_{t_p} = x \right)$$

with $\mathcal{I}(s) = \sum_{n \geq 0} 1_{[t_n, t_{n+1}]}(s) t_n$. Thus, in case **(C)**, for any function $f \in D(L)$, the mappings

$$t \in [t_{p-1}, t_p] \mapsto L_t \left(Q_{t_p, t_n}^{(m)}(f)^2 \right) \quad \text{and} \quad t \in [t_{p-1}, t_p] \mapsto L_t^2 \left(Q_{t_p, t_n}^{(m)}(f)^2 \right)$$

are uniformly bounded w.r.t. the parameter m , and differentiable with uniformly bounded derivatives w.r.t. the parameter m .

The first assertion of theorem 3.2 is based on the first order decompositions of the fluctuation of $\mu_{t_n}^N$ around its limiting value $\mu_{t_n}^{(m)}$ developed in [16]. Using theorem 6.2 in [16], we prove the following proposition.

Proposition 6.1 *For any $N \geq 1$ and any $n \in \mathbb{N}$, we have*

$$\begin{aligned}\sqrt{N} [\mu_{t_n}^N - \mu_{t_n}^{(m)}] &= \sum_{p=0}^n \frac{1}{\mu_{t_p}^N (\overline{G}_{t_p, t_n}^{(m, N)})} W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) \\ &= \sum_{p=0}^n W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) + \frac{1}{\sqrt{N}} \mathcal{R}_{t_n}^{(m, N)}(f)\end{aligned}$$

with the first order integral operator

$$D_{t_p, t_n}^{(m, N)}(f) := \overline{G}_{t_p, t_n}^{(m, N)} \left(P_{t_p, t_n}^{(m)}(f) - \Phi_{t_p, t_n}(\mu_{t_{p-1}}^N)(f) \right) \quad \text{with} \quad \overline{G}_{t_p, t_n}^{(m, N)} = G_{t_p, t_n}^{(m)} / \Phi_{t_p}^{(m)}(\mu_{t_{p-1}}^N) \left(G_{t_p, t_n}^{(m)} \right)$$

and the remainder second order term

$$\mathcal{R}_{t_n}^{(m, N)}(f) = - \sum_{p=0}^n \frac{1}{\mu_{t_p}^N (\overline{G}_{t_p, t_n}^{(m, N)})} W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, t_n}^{(m, N)} \right) W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)$$

In the above, we have used the convention $W_{t_{-1}, t_0}^N = \sqrt{N}[\mu_{t_0}^N - \mu_{t_0}]$, for $p = 0$.

We are now in position to prove the bias and the variance estimates stated in theorem 3.2 for the continuous time models.

Proof of theorem 3.2 - case (C):

Firstly, the first order decomposition stated above clearly implies that

$$N \mathbb{E} \left(\left[\mu_{t_n}^N - \mu_{t_n}^{(m)} \right] \right) = \mathbb{E} \left(\mathcal{R}_{t_n}^{(m, N)}(f) \right).$$

On the other hand, we have

$$\left| \mathbb{E} \left(\mathcal{R}_{t_n}^{(m, N)}(f) \right) \right| \leq c_{t_n}^1 \sum_{p=0}^n \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, t_n}^{(m, N)} \right)^2 \right)^{1/2} \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)^2 \right)^{1/2}.$$

To get one step further, we use the fact that

$$\begin{aligned}&\mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)^2 \right) \\ &= \mathbb{E} \left(\mu_{t_{p-1}}^N \left[\Gamma_{L_{t_{p-1}, \mu_{t_{p-1}}^N}} \left(D_{t_p, t_n}^{(m, N)}(f), D_{t_p, t_n}^{(m, N)}(f) \right) \right] \right) \frac{1}{m} + \mathbb{E} \left(R_{t_{p-1}}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) \right) \frac{1}{m^2}.\end{aligned}$$

After some elementary manipulations we prove that

$$\sup_{0 \leq p \leq n} \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)^2 \right) \leq c_{t_n}(f)/m \quad \text{and} \quad \sup_{0 \leq p \leq n} \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, t_n}^{(m, N)} \right)^2 \right) \leq c_{t_n}/m. \quad (33)$$

This ends the proof of the bias estimate.

The proof of the variance estimates is based on the following technical lemma.

Lemma 6.2 For any f with $\text{osc}(f) \leq 1$, we have the fourth conditional moment estimate

$$\mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^4 \mid \xi_{t_n} \right) \leq \frac{1}{N} \mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^2 \mid \xi_{t_n} \right) + 6 \mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^2 \mid \xi_{t_n} \right)^2.$$

Proof: With some elementary computation, we have that

$$\begin{aligned} & \mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^4 \mid \xi_{t_n} \right) \\ &= \frac{1}{N} \int \mu_{t_n}^N(du) \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u, dv) \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^4 \\ &+ 6 \left(1 - \frac{1}{N} \right) \int_{u \neq u'} \mu_{t_n}^N(du) \mu_{t_n}^N(du') \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u, dv) \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^2 \\ &\quad \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u', dv') \left(f(v') - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u') \right)^2. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E} \left(W_{t_n, t_{n+1}}^N(f)^4 \mid \xi_{t_n} \right) &\leq \frac{1}{N} \int \mu_{t_n}^N(du) \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u, dv) \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^4 \\ &+ 6 \left[\int \mu_{t_n}^N(du) \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(u, dv) \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^2 \right]^2. \end{aligned}$$

The end of the proof is based on the fact that

$$\left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^4 \leq \left(f(v) - \mathcal{K}_{t_n, t_{n+1}, \mu_{t_n}^N}(f)(u) \right)^2$$

as soon as $\text{osc}(f) \leq 1$. This ends the proof of the lemma. ■

Combining this lemma with (33), we prove that

$$\sup_{0 \leq p \leq n} \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)^4 \right) \leq c_{t_n}(f) \frac{1}{m} \left[\frac{1}{N} + \frac{1}{m} \right]$$

and

$$\sup_{0 \leq p \leq n} \mathbb{E} \left(W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, t_n}^{(m, N)} \right)^4 \right) \leq c_{t_n}(f) \frac{1}{m} \left[\frac{1}{N} + \frac{1}{m} \right].$$

This implies that

$$\mathbb{E} \left(\mathcal{R}_{t_n}^{(m, N)}(f)^2 \right)^{1/2} \leq c_{t_n} \sum_{p=0}^n \mathbb{E} \left[W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, t_n}^{(m, N)} \right)^4 \right]^{1/4} \mathbb{E} \left[W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right)^4 \right]^{1/4}$$

and therefore

$$\mathbb{E} \left(\mathcal{R}_{t_n}^{(m, N)}(f)^2 \right) \leq c_{t_n}(f) (1 + m/N).$$

Using the fact that

$$N \mathbb{E} \left(\left(\mu_{t_n}^N(f) - \mu_{t_n}^{(m)}(f) \right)^2 \right) \leq 2 \left[\mathbb{E} \left(\left(\sum_{p=0}^n W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) \right)^2 \right) + \frac{1}{N} \mathbb{E} \left(\mathcal{R}_{t_n}^{(m, N)}(f)^2 \right) \right]$$

with

$$\mathbb{E} \left(\left(\sum_{p=0}^n W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) \right)^2 \right) = \sum_{p=0}^n \mathbb{E} \left(\left(W_{t_{p-1}, t_p}^N \left(D_{t_p, t_n}^{(m, N)}(f) \right) \right)^2 \right) \leq c_{t_n}(f)$$

we conclude that

$$N \mathbb{E} \left(\left(\mu_{t_n}^N(f) - \mu_{t_n}^{(m)}(f) \right)^2 \right) \leq c_{t_n}(f) \left(1 + \frac{1}{N} + \frac{m}{N^2} \right).$$

This ends the proof of theorem 3.2. ■

6.2 Discrete time models

The main objective of this section is to prove theorem 3.2 for the discrete time models related to case **(D)**. In this situation, we recall that $\mu_{t_{nm}} = \eta_n$, for any integer parameter $n \in \mathbb{N}$. As usual, the approximation measures $\mu_{t_n}^N$ are defined as the occupation measures of the mean field particle model of the McKean distribution flow μ_{t_n} . By lemma 5.3, for any $k \in \mathbb{N}$ we have

$$L_{k-\frac{1}{m}, \mu_{k-\frac{1}{m}}}^{(m)} := \mathcal{K}_{k-\frac{1}{m}, k, \mu_{k-\frac{1}{m}}} - Id = \mathcal{S}_{k-\frac{1}{m}, \mu_{k-\frac{1}{m}}} M_k - Id$$

and for any $(k-1)m < p < km$, with $k \in \mathbb{N}$, we have

$$\mathcal{M}_{t_{p-1}, t_p} = Id \implies L_{t_{p-1}, \mu_{t_{p-1}}}^{(m)} = \mathcal{S}_{t_{p-1}, \mu_{t_{p-1}}} - Id.$$

When the Markov transport equation (5) is met for some Markov transitions $\mathcal{S}_{t_n, \mu}$ satisfying the first order decomposition (6), we have for any $k \in \mathbb{N}$ and any $(k-1)m < p < km$

$$L_{t_{p-1}, \mu_{t_{p-1}}}^{(m)} = \widehat{L}_{t_{p-1}, \mu_{t_{p-1}}} \frac{1}{m} + \widehat{R}_{t_{p-1}, \mu_{t_{p-1}}} \frac{1}{m^2}.$$

Using the same line of argument as in (32), we prove that

$$\begin{aligned} \mathbb{E} \left(W_{t_{p-1}, t_p}^N(f)^2 \mid \xi_{t_n} \right) &= \mu_{t_{p-1}}^N \left[\Gamma_{L_{t_{p-1}, \mu_{t_{p-1}}}^{(m)}}(f, f) \right] - \mu_{t_{p-1}}^N \left(\left(L_{t_{p-1}, \mu_{t_{p-1}}}^{(m)}(f) \right)^2 \right) \\ &= \mu_{t_{p-1}}^N \left[\Gamma_{\widehat{L}_{t_{p-1}, \mu_{t_{p-1}}}^{(m)}}(f, f) \right] \frac{1}{m} + R_{t_{p-1}}^N(f) \frac{1}{m^2} \end{aligned}$$

with some remainder term $R_{t_{p-1}}^N(f)$ such that

$$\sup_{N \geq 1} \left| R_{t_{p-1}}^N(f) \right| \leq c \left\| \mathcal{U}_{t_{p-1}} \right\|^2 \text{osc}(f)^2.$$

This clearly implies that

$$\mathbb{E} \left(W_{t_{p-1}, t_p}^N(f)^2 \mid \xi_{t_n} \right) \leq c \left\| \mathcal{U}_{t_{p-1}} \right\|^2 \text{osc}(f)^2 \frac{1}{m} \quad \text{and} \quad \mathbb{E} \left(W_{k-\frac{1}{m}, k}^N(f)^2 \mid \xi_{k-\frac{1}{m}} \right) \leq \text{osc}(f)^2. \quad (34)$$

As in proposition 6.1, we prove the following decomposition.

Proposition 6.3 *For any $N \geq 1$, $f \in \mathcal{B}_b(E)$, and any $n \in \mathbb{N}$ we have the decomposition*

$$[\mu_n^N - \eta_n](f) = A_n^N + B_n^N$$

with

$$A_n^N := \sum_{1 \leq k \leq n} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \frac{1}{\mu_{t_p}^N \left(\overline{G}_{t_p, n}^{(m, N)} \right)} W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right)$$

and

$$B_n^N = \sum_{0 \leq k \leq n} \frac{1}{\mu_k^N \left(\overline{G}_{k, n}^{(m, N)} \right)} W_{k-\frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right).$$

In the above, the function $\overline{G}_{t_p, n}^{(m, N)}$ and the integral operator $D_{t_p, n}^{(m, N)}(f)$ are defined in proposition 6.1, and for $k = 0$, we have used the convention $W_{-\frac{1}{m}, 0}^N = W_0^N$.

To analyze the bias and the variance, we also need to consider the first order decompositions presented in the following corollary.

Corollary 6.4 *For any $N \geq 1$ and any $n \in \mathbb{N}$ we have the decomposition*

$$A_n^N = A_n^{(N, 1)} + \frac{1}{\sqrt{N}} A_n^{(N, 2)} \quad \text{and} \quad B_n^N = B_n^{(N, 1)} + \frac{1}{\sqrt{N}} B_n^{(N, 2)}$$

with the first order terms

$$\begin{aligned} A_n^{(N, 1)} &= \sum_{1 \leq k \leq n} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \\ B_n^{(N, 1)} &= \sum_{0 \leq k \leq n} W_{k-\frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \end{aligned}$$

and the remainder second order terms

$$\begin{aligned} A_n^{(N, 2)} &= - \sum_{1 \leq k \leq n} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \frac{1}{\mu_{t_p}^N \left(\overline{G}_{t_p, n}^{(m, N)} \right)} W_{t_{p-1}, t_p}^N \left(\overline{G}_{t_p, n}^{(m, N)} \right) W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \\ B_n^{(N, 2)} &= - \sum_{0 \leq k \leq n} \frac{1}{\mu_k^N \left(\overline{G}_{k, n}^{(m, N)} \right)} W_{k-\frac{1}{m}, k}^N \left(\overline{G}_{k, n}^{(m, N)} \right) W_{k-\frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \end{aligned}$$

Now we come to the proof of the bias and the variance estimates presented in theorem 3.2.

In the further development of this section f stands for some bounded function s.t. $\text{osc}(f) \leq 1$.

By construction, the random fields W_{t_{p-1}, t_p}^N and W_{t_{q-1}, t_q}^N are uncorrelated for any $p \neq q$. Combining this property with the estimates (34) we prove that

$$\mathbb{E} \left(\left(A_n^{(N,1)} \right)^2 \right) = \sum_{1 \leq k \leq n} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \mathbb{E} \left(\left[W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \right]^2 \right).$$

On the other hand, we have

$$\mathbb{E} \left(\left[W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \right]^2 \right) \leq \frac{c}{m} \left\| \mathcal{U}_{t_{p-1}} \right\|^2 \left(g_{t_p, n}^{(m)} \text{osc} \left(P_{t_p, n}^{(m)}(f) \right) \right)^2$$

with

$$g_{t_p, n}^{(m)} := \sup_{x, y} \left[G_{t_p, n}^{(m)}(x) / G_{t_p, n}^{(m)}(y) \right].$$

We prove the last assertion using the fact that

$$\text{osc} \left(D_{t_p, n}^{(m, N)}(f) \right) / 2 \leq \left\| D_{t_p, n}^{(m, N)}(f) \right\| \leq g_{t_p, n}^{(m)} \text{osc} \left(P_{t_p, n}^{(m)}(f) \right).$$

By the semigroup formulae (9), for any $p = (k-1)m + r$, with $r < m$ we find that

$$P_{t_p, n}^{(m)} = P_{(k-1), n} \quad \text{and} \quad g_{t_p, n}^{(m)} \leq g_{k-1, n} \times \sup_{x, y} \left(\frac{G_{k-1}(x)}{G_{k-1}(y)} \right)^{1/m} = g_{k-1, n} \times g_{k-1, k}^{1/m}$$

with

$$g_{k-1, n} := \sup_{x, y} \left[Q_{(k-1), n}(1)(x) / Q_{(k-1), n}(1)(y) \right].$$

This implies that

$$\mathbb{E} \left(\left(A_n^{(N,1)} \right)^2 \right) \leq c \sum_{1 \leq k \leq n} \left\| \log G_{k-1} \right\|^2 \left(g_{k-1, n} \times g_{k-1, k}^{1/m} \text{osc} \left(P_{(k-1), n}(f) \right) \right)^2.$$

In the same way, we prove that

$$\begin{aligned} \mathbb{E} \left(\left(B_n^{(N,1)} \right)^2 \right) &= \sum_{0 \leq k \leq n} \mathbb{E} \left(\left[W_{k-\frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \right]^2 \right) \\ &\leq c \sum_{0 \leq k \leq n} \left[g_{k, n}^{(m)} \text{osc} \left(P_{k, n}^{(m)}(f) \right) \right]^2 = c \sum_{0 \leq k \leq n} \left[g_{k, n} \text{osc} \left(P_{k, n}(f) \right) \right]^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
& \left| \mathbb{E} \left(A_n^{(N,2)} \right) \right| \\
& \leq \sum_{1 \leq k \leq n} g_{k-1,n} g_{k-1,k}^{1/m} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \\
& \quad \mathbb{E} \left(\left(W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right)^2 \right)^{1/2} \mathbb{E} \left(\left(W_{t_{p-1},t_p}^N \left(D_{t_p,n}^{(m,N)}(f) \right) \right)^2 \right)^{1/2} \\
& \leq \sum_{0 \leq k < n} \|\log G_k\|^2 (g_{k,n} g_{k,k+1})^3 \text{osc}(P_{k,n}(f)).
\end{aligned} \tag{35}$$

Arguing as in the end of the proof of theorem 3.2, we can also check that

$$\begin{aligned}
& \mathbb{E} \left(\left(A_n^{(N,2)} \right)^2 \right)^{1/2} \\
& \leq \sum_{1 \leq k \leq n} g_{k-1,n} g_{k-1,k}^{1/m} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \mathbb{E} \left(\left(W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) W_{t_{p-1},t_p}^N \left(D_{t_p,n}^{(m,N)}(f) \right) \right)^2 \right)^{1/2} \\
& \leq \sum_{1 \leq k \leq n} g_{k-1,n} g_{k-1,k}^{1/m} \sum_{p=(k-1)m+1}^{(k-1)m+(m-1)} \\
& \quad \mathbb{E} \left(\left(W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right)^4 \right)^{1/4} \mathbb{E} \left(\left(W_{t_{p-1},t_p}^N \left(D_{t_p,n}^{(m,N)}(f) \right) \right)^4 \right)^{1/4}.
\end{aligned}$$

On the other hand, using lemma 6.2 we have

$$\begin{aligned}
& \mathbb{E} \left(\left[W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right]^4 \right) \\
& \leq \frac{1}{N} (2g_{t_p,n}^{(m)})^2 \mathbb{E} \left(\left[W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right]^2 \right) + 6 \mathbb{E} \left(\left[W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right]^2 \right)^2 \\
& \leq c (g_{t_p,n}^{(m)})^4 \left(\frac{1}{N} \|\mathcal{U}_{t_{p-1}}\|^2 \frac{1}{m} + \|\mathcal{U}_{t_{p-1}}\|^4 \frac{1}{m^2} \right)
\end{aligned}$$

from which we find the crude upper bound

$$\mathbb{E} \left(\left[W_{t_{p-1},t_p}^N \left(\overline{G}_{t_p,n}^{(m,N)} \right) \right]^4 \right) \leq c g_{k-1,n}^4 g_{k-1,k}^4 (\|\log G_{k-1}\| \vee 1)^4 / m^2.$$

for any $N \geq m$, and for any $p = (k-1)m + r$, with $r < m$. Using the same line of arguments, we have

$$\begin{aligned} & \mathbb{E} \left(\left[W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \right]^4 \right) \\ & \leq \frac{1}{N} \left(2 g_{t_p, n}^{(m)} \text{osc} \left(P_{t_p, n}^{(m)}(f) \right) \right)^2 \mathbb{E} \left(\left[W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \right]^2 \right) + 6 \mathbb{E} \left(\left[W_{t_{p-1}, t_p}^N \left(D_{t_p, n}^{(m, N)}(f) \right) \right]^2 \right)^2 \\ & \leq c g_{k-1, n}^4 g_{k-1, k}^{4/m} \text{osc} \left(P_{(k-1), n}(f) \right)^4 (\|\log G_{k-1}\| \vee 1)^4 / m^2 \end{aligned}$$

for any $N \geq m$. This implies that

$$\mathbb{E} \left(\left(A_n^{(N, 2)} \right)^2 \right)^{1/2} \leq c \sum_{1 \leq k \leq n} g_{k-1, n}^3 g_{k-1, k}^3 (\|\log G_{k-1}\| \vee 1)^2 \text{osc} \left(P_{(k-1), n}(f) \right)$$

for any $N \geq m$. One concludes that

$$\mathbb{E} \left((A_n^N)^2 \right) \leq c \left(a_{1, n} + \frac{1}{N} a_{2, n}^2 \right) \leq c a_{2, n} \left(1 + \frac{1}{N} a_{2, n} \right)$$

with

$$a_{1, n} := \sum_{0 \leq k < n} \|\log G_k\|^2 (g_{k, n} g_{k, k+1} \text{osc} (P_{k, n}(f)))^2 \leq a_{2, n}$$

and

$$a_{2, n} := \sum_{0 \leq k < n} g_{k, n}^3 g_{k, k+1}^3 (\|\log G_k\| \vee 1)^2 \text{osc} (P_{k, n}(f)).$$

In much the same way, we prove that

$$\mathbb{E} \left(\left(B_n^{(N, 2)} \right)^2 \right)^{1/2} \leq \sum_{0 \leq k \leq n} g_{k, n} \mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(\overline{G}_{k, n}^{(m, N)} \right) \right]^4 \right)^{1/4} \mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \right]^4 \right)^{1/4}.$$

Using the fact that

$$\mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(\overline{G}_{k, n}^{(m, N)} \right) \right]^2 \right) \leq c g_{k, n}^2$$

and

$$\mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \right]^2 \right) \leq c g_{k, n}^2 \text{osc} (P_{k, n}(f))^2$$

we check that

$$\begin{aligned} \left| \mathbb{E} \left(B_n^{(N, 2)} \right) \right| & \leq \sum_{0 \leq k \leq n} g_{k, n} \mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(\overline{G}_{k, n}^{(m, N)} \right) \right]^2 \right)^{1/2} \mathbb{E} \left(\left[W_{k - \frac{1}{m}, k}^N \left(D_{k, n}^{(m, N)}(f) \right) \right]^2 \right)^{1/2} \\ & \leq c \sum_{0 \leq k \leq n} g_{k, n}^3 \text{osc} (P_{k, n}(f)). \end{aligned} \tag{36}$$

Combining (35) with (36), we obtain the bias estimate

$$\begin{aligned}
& N \left| \mathbb{E} \left(\mu_n^N(f) \right) - \eta_n(f) \right| \\
& \leq c \left(\sum_{0 \leq k \leq n} g_{k,n}^3 \operatorname{osc} (P_{k,n}(f)) + \sum_{0 \leq k < n} \|\log G_k\|^2 (g_{k,n} g_{k,k+1})^3 \operatorname{osc} (P_{k,n}(f)) \right) \\
& \leq c \sum_{0 \leq k \leq n} (\|\log G_k\| \vee 1)^2 (g_{k,n} g_{k,k+1})^3 \operatorname{osc} (P_{k,n}(f)).
\end{aligned}$$

In addition, we have that

$$\begin{aligned}
& \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(\overline{G}_{k,n}^{(m,N)} \right) \right]^4 \right) \\
& \leq c \left[\frac{g_{k,n}^2}{N} \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(\overline{G}_{k,n}^{(m,N)} \right) \right]^2 \right) + \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(\overline{G}_{k,n}^{(m,N)} \right) \right]^2 \right)^2 \right] \leq c g_{k,n}^4
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(D_{k,n}^{(m,N)}(f) \right) \right]^4 \right) \\
& \leq c \left[\frac{g_{k,n}^2}{N} \operatorname{osc} (P_{k,n}(f))^2 \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(D_{k,n}^{(m,N)}(f) \right) \right]^2 \right) + \mathbb{E} \left(\left[W_{k-\frac{1}{m},k}^N \left(D_{k,n}^{(m,N)}(f) \right) \right]^2 \right)^2 \right] \\
& \leq c g_{k,n}^4 \operatorname{osc} (P_{k,n}(f))^4.
\end{aligned}$$

This implies that

$$\mathbb{E} \left(\left(B_n^{(N,2)} \right)^2 \right)^{1/2} \leq c \sum_{0 \leq k \leq n} g_{k,n}^3 \operatorname{osc} (P_{k,n}(f)).$$

We conclude that

$$\mathbb{E} \left((B_n^N)^2 \right) \leq c \left(b_{1,n} + \frac{1}{N} b_{2,n}^2 \right) \leq c b_{2,n} \left(1 + \frac{1}{N} b_{2,n} \right)$$

with

$$b_{1,n} := \sum_{0 \leq k \leq n} [g_{k,n} \operatorname{osc} (P_{k,n}(f))]^2 \leq b_{2,n} \leq a_{2,n} \quad \text{and} \quad b_{2,n} := \sum_{0 \leq k \leq n} g_{k,n}^3 \operatorname{osc} (P_{k,n}(f)).$$

This yields the variance estimate

$$N \mathbb{E} \left(\left(\mu_n^N(f) - \eta_n(f) \right)^2 \right) \leq c a_{2,n} \left(1 + \frac{1}{N} a_{2,n} \right).$$

This ends the proof of the theorem. ■

7 Appendix

7.1 Proof of theorem 3.1

We let $\underline{\tau}$ and $\overline{\tau}$ be the mappings on \mathbb{R}_+ defined by

$$\underline{\tau}(s) = \sum_{n \geq 0} 1_{[t_n, t_{n+1}[}(s) t_n \quad \text{and} \quad \overline{\tau}(s) = \sum_{n \geq 0} 1_{[t_n, t_{n+1}[}(s) t_{n+1}.$$

With this notation, we clearly have that

$$\int_0^{t_n} \mathcal{V}_{\underline{\tau}(s)}(\mathcal{X}_{\underline{\tau}(s)}) ds = \sum_{0 \leq p < n} \int_{t_p}^{t_{p+1}} \mathcal{V}_{\underline{\tau}(s)}(\mathcal{X}_{\underline{\tau}(s)}) ds = \sum_{0 \leq p < n} \mathcal{V}_{t_p}(\mathcal{X}_{t_p})/m.$$

In case **(D)** we readily check that $\gamma_{\lfloor t \rfloor} = \nu_{\lfloor t \rfloor}$, for any $t \in \mathbb{R}_+$. More precisely, we have

$$n = km + r \quad \text{with} \quad k \geq 0, \quad 0 \leq r < m \implies t_n = k + r/m$$

and the pair of processes $(\mathcal{V}_s, \mathcal{X}_s)$ only change at integer times, that is we have that

$$t_{pm} = p \leq s < t_{(p+1)m} = (p+1) \implies \mathcal{V}_s = \log G_p \quad \text{and} \quad \mathcal{X}_s = X_p$$

so that for any $q \in \mathbb{N}$ we have that

$$t_q \leq s < t_{q+1} \implies \mathcal{V}_s = \mathcal{V}_{t_q} \quad \text{and} \quad \mathcal{X}_s = \mathcal{X}_{t_q}.$$

Furthermore, using the fact that

$$\int_0^{t_n} \mathcal{V}_s(\mathcal{X}_s) ds = \sum_{0 \leq p < n} \mathcal{V}_{t_p}(\mathcal{X}_{t_p})/m$$

we readily check that

$$\nu_{t_n}(f) = \mathbb{E} \left(f(\mathcal{X}_{t_n}) \prod_{0 \leq p < n} e^{\mathcal{V}_{t_p}(\mathcal{X}_{t_p})/m} \right) \quad \nu_{t_{nm}} = \gamma_n \quad \text{and} \quad \mu_{t_{nm}} = \eta_n.$$

Now we come to the case **(C)**. We have the Ito formula

$$d\mathcal{V}_t(\mathcal{X}_t) = \left(\frac{\partial}{\partial t} + L_t \right) (\mathcal{V}_t)(\mathcal{X}_t) + dM_t(\mathcal{V})$$

with a martingale term $M_t(\mathcal{V})$ with predictable angle bracket

$$d\langle M(\mathcal{V}) \rangle_t = \Gamma_{L_t}(\mathcal{V}_t, \mathcal{V}_t)(\mathcal{X}_t) dt$$

defined in terms of the carré du champ Γ_{L_t} operator associated with the generator L_t and defined for any $f \in D(L)$ by the following formula

$$\Gamma_{L_t}(f, f)(x) = L_t((f - f(x))^2)(x) = L_t(f^2)(x) - 2f(x)L_t(f)(x).$$

We recall that the predictable process $\langle M(\mathcal{V}) \rangle_t$ is the unique right-continuous and increasing predictable such that the random process $M_t(\mathcal{V})^2 - \langle M(\mathcal{V}) \rangle_t$ is again a martingale. We also recall that $M_t(\mathcal{V})^2 - [M(\mathcal{V})]_t$ is also a martingale, for the quadratic variation $[M(\mathcal{V})]$ of the process $M(\mathcal{V})$ defined as

$$[M(\mathcal{V})]_t = \lim_{\|\pi\| \rightarrow 0} \sum_{k=1}^n (M_{t_k}(\mathcal{V}) - M_{t_{k-1}}(\mathcal{V}))^2$$

where π ranges over partitions of the interval $[0, t]$, and the norm $\|\pi\|$ of the partition π is the size of the mesh. For continuous martingales $M_t(\mathcal{V})$, it is well known that $[M(\mathcal{V})] = \langle M(\mathcal{V}) \rangle$.

Using an elementary integration by part formula, for any $p \geq 0$ we have

$$\int_{t_p}^{t_{p+1}} (\mathcal{V}_s(\mathcal{X}_s) - \mathcal{V}_{t_p}(\mathcal{X}_{t_p})) ds = \int_{t_p}^{t_{p+1}} (t_{p+1} - s) d\mathcal{V}_s(\mathcal{X}_s)$$

from which we conclude that

$$\int_0^{t_n} \mathcal{V}_s(\mathcal{X}_s) ds = \int_0^{t_n} \mathcal{V}_{\tau(s)}(\mathcal{X}_{\tau(s)}) ds + \int_0^{t_n} (\bar{\tau}(s) - s) d\mathcal{V}_s(\mathcal{X}_s).$$

We set

$$R_{t_n}(f) = \mathbb{E} \left(f(\mathcal{X}_{[0, t_n]}) \left[e^{\int_0^{t_n} \mathcal{V}_s(\mathcal{X}_s) ds} - e^{\int_0^{t_n} \mathcal{V}_{\tau(s)}(\mathcal{X}_{\tau(s)}) ds} \right] \right).$$

Using the fact that $|e^x - e^y| \leq |x - y| |e^x + e^y|$, we prove that

$$|R_{t_n}(f)| \leq \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) d\mathcal{V}_s(\mathcal{X}_s) \right] \left[e^{\int_0^{t_n} \mathcal{V}_s(\mathcal{X}_s) ds} + e^{\int_0^{t_n} \mathcal{V}_{\tau(s)}(\mathcal{X}_{\tau(s)}) ds} \right] \right).$$

Using Hölder's inequality we have

$$|R_{t_n}(f)| \leq \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) d\mathcal{V}_s(\mathcal{X}_s) \right]^p \right)^{1/p} \mathbb{E} \left(\left[e^{\int_0^{t_n} \mathcal{V}_s(\mathcal{X}_s) ds} + e^{\int_0^{t_n} \mathcal{V}_{\tau(s)}(\mathcal{X}_{\tau(s)}) ds} \right]^{p'} \right)^{1/p'}.$$

To estimate the first term in the r.h.s. of the above estimate, we use the inequality

$$\begin{aligned} & \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) d\mathcal{V}_s(\mathcal{X}_s) \right]^p \right)^{1/p} \\ & \leq \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) \left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s)(\mathcal{X}_s) ds \right]^p \right)^{1/p} + \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) dM_s(\mathcal{V}) \right]^p \right)^{1/p}. \end{aligned}$$

Using the generalized Minkowski inequality we prove that

$$\begin{aligned} & \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) \left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s)(\mathcal{X}_s) ds \right]^p \right)^{1/p} \\ & \leq \int_0^{t_n} (\bar{\tau}(s) - s) \mathbb{E} \left(\left[\left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s)(\mathcal{X}_s) \right]^p \right)^{1/p} ds \\ & \leq \frac{1}{m} \int_0^{t_n} \mathbb{E} \left(\left[\left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s)(\mathcal{X}_s) \right]^p \right)^{1/p} ds. \end{aligned}$$

On the other hand, by the Burkholder-Davis-Gundy inequality, for any $p > 0$ we have that

$$\begin{aligned} \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s) dM_s(\mathcal{V}) \right]^p \right)^{1/p} &\leq c_p \mathbb{E} \left(\left[\int_0^{t_n} (\bar{\tau}(s) - s)^2 d[M(\mathcal{V})]_s \right]^{p/2} \right)^{1/p} \\ &\leq \frac{c_p}{m} \mathbb{E} \left([M(\mathcal{V})]_{t_n}^{p/2} \right)^{1/p} \end{aligned}$$

for some finite constant $c_p < \infty$ whose values only depends on the parameter p . The end of the proof of the first assertion. The second one is proved using the decomposition

$$\left[\mathbb{Q}_{t_n}^{(m)} - \mathbb{Q}_{t_n} \right] (f) = \frac{1}{\Lambda_{t_n}^{(m)}(1)} \left[\Lambda_{t_n}^{(m)}(f_n) - \Lambda_{t_n}(f_n) \right]$$

with the centered function $f_n = (f - \Lambda_{t_n}(f))$.

$$\left| \mathbb{Q}_{t_n}^{(m)}(f) - \mathbb{Q}_{t_n}(f) \right| \leq \frac{2}{\Lambda_{t_n}^{(m)}(1) \vee \Lambda_{t_n}(1)} \sup_{f \in \mathcal{B}_b(E_n) : \|f\| \leq 1} \left| \Lambda_{t_n}^{(m)}(f) - \Lambda_{t_n}(f) \right|.$$

From these estimates, we find that

$$\|\bar{r}_{m,t_n}\|_{\text{tv}} \leq \frac{2}{\Lambda_{t_n}^{(m)}(1) \vee \Lambda_{t_n}(1)} \|r_{m,t_n}\|_{\text{tv}}$$

and

$$\sup_{m \geq 1} \|r_{m,t_n}\|_{\text{tv}} \leq a_p b_{t_n}^{(p')} \left(c_{t_n}^{(p)} + d_{t_n}^{(p)} \right)$$

with for any $\frac{1}{p} + \frac{1}{p'} = 1$

$$b_t^{(p')} = \mathbb{E} \left(e^{p' \int_0^t \mathcal{V}_s(\mathcal{X}_s) ds} \right)^{1/p'} \vee \mathbb{E} \left(e^{p' \int_0^t \mathcal{V}_{\mathbb{Z}(s)}(\mathcal{X}_{\mathbb{Z}(s)}) ds} \right)^{1/p'} < \infty$$

as well as

$$c_t^{(p)} := \int_0^t \mathbb{E} \left(\left[\left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s)(\mathcal{X}_s) \right]^p \right)^{1/p} ds < \infty \quad \text{and} \quad d_t^{(p)} := \mathbb{E} \left([M(\mathcal{V})]_t^{p/2} \right)^{1/p} < \infty.$$

Whenever $\mathcal{V} \in \mathcal{C}^1([0, \infty[, \mathcal{D}(L))$, we have

$$b_t^{(p')} \leq \exp \left(t \sup_{0 \leq s \leq t} \|\mathcal{V}_s\| \right) \quad \text{and} \quad c_t^{(p)} \leq t \sup_{0 \leq s \leq t} \left\| \left(\frac{\partial}{\partial s} + L_s \right) (\mathcal{V}_s) \right\|.$$

Notice that for stochastic processes with continuous paths, the constant $d_t^{(p)}$ is given by

$$d_t^{(p)} := \mathbb{E} \left(\langle M(\mathcal{V}) \rangle_t^{p/2} \right)^{1/p} = \mathbb{E} \left(\left(\int_0^t \Gamma_{L_s}(\mathcal{V}_s, \mathcal{V}_s)(\mathcal{X}_s) ds \right)^{p/2} \right)^{1/p} \leq \sqrt{t} \sup_{0 \leq s \leq t} \|\Gamma_{L_s}(\mathcal{V}_s, \mathcal{V}_s)\|^{1/2}.$$

This ends the proof of the theorem. ■

7.2 Proof of proposition 5.3

In case **(D)**, the mutation transition only occurs at integer times. More formally, we have $\mathcal{X}_{t_n} = X_{[t_n]}$, so that for any

$$n = km + r \quad \text{with} \quad r + 1 < m$$

we have

$$t_n = k + r/m \Rightarrow \mathcal{M}_{t_n, t_{n+1}}(y, dz) = \mathcal{M}_{k+r/m, k+(r+1)/m}(y, dz) = \delta_y(dz).$$

On the other hand, we have that

$$n = km + m - 1 \Rightarrow t_n = (k + 1) - 1/m \quad \text{with} \quad t_{n+1} = (k + 1)$$

and

$$\mathcal{M}_{t_n, t_{n+1}}(y, dz) = \mathcal{M}_{(k+1)-1/m, (k+1)}(y, dz) = M_{k+1}(y, dz).$$

This ends the proof of the first assertion. Now, we come to the proof of (24). Firstly, we recall that

$$f(t_{n+1}, X_{t_{n+1}}) = f(t_n, X_{t_n}) + \int_{t_n}^{t_{n+1}} \left(\frac{\partial}{\partial s} + L_s \right) (f)(s, X_s) ds + M_{t_{n+1}}(f) - M_{t_n}(f)$$

for any $f \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$, with some martingale $M_t(f)$. This implies that

$$\mathbb{E}_{t_n, x}(f(t_{n+1}, X_{t_{n+1}})) = f(t_n, x) + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n, x} \left[\left(\frac{\partial}{\partial s} + L_s \right) (f)(s, X_s) \right] ds$$

where $\mathbb{E}_{t_n, x}(\cdot)$ stands for the conditional expectation operator given that $X_{t_n} = x$. Iterating this formula, we find that

$$\mathbb{E}_{t_n, x} \left[\left(\frac{\partial}{\partial s} + L_s \right) (f)(s, X_s) \right] = \left(\frac{\partial}{\partial t_n} + L_{t_n} \right) (f)(t_n, x) + \int_{t_n}^s \mathbb{E}_{t_n, x} \left[\left(\frac{\partial}{\partial r} + L_r \right)^2 (f)(r, X_r) \right] dr$$

as soon as $\left(\frac{\partial}{\partial t} + L_t \right) (f) \in \mathcal{C}^1([t_n, t_{n+1}], D(L))$. Under this condition, we find the first order decomposition

$$\mathbb{E}_{t_n, x}(f(t_{n+1}, X_{t_{n+1}})) = f(t_n, x) + \left(\frac{\partial}{\partial t_n} + L_{t_n} \right) (f)(t_n, x) \frac{1}{m} + R_{t_n}(f) \frac{1}{m^2}$$

with some remainder operator

$$R_{t_n}(f) = m^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathbb{E}_{t_n, x} \left[\left(\frac{\partial}{\partial r} + L_r \right)^2 (f)(r, X_r) \right] dr$$

such that

$$\|R_{t_n}(f)\| \leq \sup_{x \in E} \sup_{t_n \leq t < t_{n+1}} \left| \mathbb{E}_{t_n, x} \left[\left(\frac{\partial}{\partial t} + L_t \right)^2 (f)(t, X_t) \right] \right|.$$

The end of the proof of (24) is now clear. Using lemma 5.2, and the first order decompositions (37) and (24) we have

$$L_{t_n, \mu}^{(m)} = L_{t_n, \mu} \frac{1}{m} + R_{t_n, \mu} \frac{1}{m^2}$$

with

$$R_{t_n, \mu} = R_{t_n} + \widehat{R}_{t_n, \mu} + m^2 \left[\widehat{L}_{t_n, \mu} \frac{1}{m} + \widehat{R}_{t_n, \mu} \frac{1}{m^2} \right] \left[L_{t_n} \frac{1}{m} + R_{t_n} \frac{1}{m^2} \right].$$

On the other hand, we have the estimates

$$m^2 \left[\widehat{L}_{t_n, \mu} \frac{1}{m} + \widehat{R}_{t_n, \mu} \frac{1}{m^2} \right] \left[L_{t_n} \frac{1}{m} + R_{t_n} \frac{1}{m^2} \right] = \widehat{L}_{t_n, \mu} \left[L_{t_n} + R_{t_n} \frac{1}{m} \right] + \widehat{R}_{t_n, \mu} L_{t_n}^{(m)}$$

with

$$\left\| \widehat{R}_{t_n, \mu} L_{t_n}^{(m)}(f) \right\| \leq c_{t_n} \|f\|$$

and

$$\begin{aligned} \left\| \widehat{L}_{t_n, \mu} \left[L_{t_n} + R_{t_n} \frac{1}{m} \right] (f) \right\| &\leq c_{t_n} \left\| L_{t_n}(f) + R_{t_n}(f) \frac{1}{m} \right\| \\ &\leq c_{t_n} \sup_{t_n \leq t \leq t_{n+1}} (\|\partial L_t(f)/\partial t\| + \|L_t(f)\| + \|L_t^2(f)\|) \end{aligned}$$

for some finite constant $c_{t_n} < \infty$. This ends the proof of (25).

The proof of the last assertion is based on the decomposition

$$\mathcal{K}_{t_n, t_{n+1}, \mu} \left([f - \mathcal{K}_{t_n, t_{n+1}, \mu}(f)(x)]^2 \right) (x) = \Gamma_{L_{t_n, \mu}^{(m)}}(f, f)(x) - \left(L_{t_n, \mu}^{(m)}(f)(x) \right)^2$$

and the fact that

$$\Gamma_{L_{t_n, \mu}^{(m)}}(f, f)(x) = \Gamma_{L_{t_n, \mu}}(f, f)(x) \frac{1}{m} + R_{t_n, \mu} ((f - f(x))^2)(x) \frac{1}{m^2}.$$

This ends the proof of the proposition. ■

7.3 Proof of the first order decompositions (6)

- **Case 1 :** We assume that $\mathcal{V}_t = -\mathcal{U}_t$, for some non negative and bounded function \mathcal{U}_t . In this situation, (5) is satisfied by the Markov transition

$$\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(x, dy) := e^{-\mathcal{U}_{t_n}(x)/m} \delta_x(dy) + \left(1 - e^{-\mathcal{U}_{t_n}(x)/m} \right) \Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})}(dy).$$

In this situation, we notice that

$$\begin{aligned} m \left[\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(f)(x) - f(x) \right] &= m \left(1 - e^{-\mathcal{U}_{t_n}(x)/m} \right) \left[\Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})}(f) - f(x) \right] \\ &= \mathcal{U}_{t_n}(x) \int (f(y) - f(x)) \mu_{t_n}^{(m)}(dy) + O(1) = \widehat{L}_{t_n, \mu_{t_n}^{(m)}}(f) + O(1). \end{aligned}$$

Much more is true; if we set

$$\begin{aligned}\widehat{R}_{t_n, \mu_{t_n}^{(m)}}(f) &:= m \left[m \left[\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(f) - f \right] - \widehat{L}_{t_n, \mu_{t_n}^{(m)}}(f) \right] \\ &= m \left[m \left(1 - e^{-\mathcal{U}_{t_n}/m} \right) \left[\Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})}(f) - f \right] - \mathcal{U}_{t_n} \left[\mu_{t_n}^{(m)}(f) - f \right] \right]\end{aligned}$$

then we find that

$$\begin{aligned}\left| \widehat{R}_{t_n, \mu_{t_n}^{(m)}}(f) \right| \\ \leq m \left| m \left(1 - e^{-\mathcal{U}_{t_n}/m} \right) - \mathcal{U}_{t_n} \right| \times \left| \Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})}(f) - f \right| + \mathcal{U}_{t_n} m \left| \Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})}(f) - \mu_{t_n}^{(m)}(f) \right|\end{aligned}$$

Using the fact that

$$m \left[\Psi_{e^{-\mathcal{U}_{t_n}/m}(\mu_{t_n}^{(m)})} - \mu_{t_n}^{(m)} \right](f) = \frac{1}{\mu_{t_n}^{(m)}(e^{-\mathcal{U}_{t_n}/m})} \mu_{t_n}^{(m)} \left(m \left[e^{-\mathcal{U}_{t_n}/m} - 1 \right] \left[f - \mu_{t_n}^{(m)}(f) \right] \right)$$

we prove the following first order expansion

$$\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(f) - f = \widehat{L}_{t_n, \mu_{t_n}^{(m)}}(f) \frac{1}{m} + \widehat{R}_{t_n, \mu_{t_n}^{(m)}}(f) \frac{1}{m^2} \quad (37)$$

with some integral operator $\widehat{R}_{t_n, \mu_{t_n}^{(m)}}$ such that

$$\sup_{m \geq 1} \left\| \widehat{R}_{t_n, \mu_{t_n}^{(m)}}(f) \right\| \leq c \|\mathcal{U}_{t_n}\|^2 \text{osc}(f).$$

- **Case 2 :** We assume that \mathcal{V}_t is a positive and bounded function. In this situation, (5) is satisfied by the Markov transition

$$\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(x, dy) := \frac{1}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \delta_x(dy) + \left(1 - \frac{1}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \right) \Psi_{(e^{\mathcal{V}_{t_n}/m-1})(\mu_{t_n}^{(m)})}(dy).$$

In this situation, we notice that

$$\begin{aligned}m \left[\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(f)(x) - f(x) \right] &= m \left(1 - \frac{1}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \right) \left[\Psi_{(e^{\mathcal{V}_{t_n}/m-1})(\mu_{t_n}^{(m)})}(f) - f(x) \right] \\ &= \int (f(y) - f(x)) \mathcal{V}_{t_n}(y) \mu_{t_n}^{(m)}(dy) + O(1).\end{aligned}$$

Using some elementary calculations, we also prove a first order expansion of the same form as in (37).

- **Case 3:** The Markov transport equation (5) is satisfied by the transitions

$$\begin{aligned} \mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(x, dy) := & \left(1 - \frac{\mu_{t_n}^{(m)} \left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+ \right)}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \right) \delta_x(dy) \\ & + \frac{\mu_{t_n}^{(m)} \left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+ \right)}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \Psi_{(e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+}(\mu_{t_n}^{(m)})(dy). \end{aligned}$$

The above Markov transition is well defined since we have

$$\frac{\mu_{t_n}^{(m)} \left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m}) 1_{\mathcal{V}_{t_n} > \mathcal{V}_{t_n}(x)} \right)}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m} 1_{\mathcal{V}_{t_n} > \mathcal{V}_{t_n}(x)}) + \mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m} 1_{\mathcal{V}_{t_n} \leq \mathcal{V}_{t_n}(x)})} \leq 1$$

as soon as

$$\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m} 1_{\mathcal{V}_{t_n} \leq \mathcal{V}_{t_n}(x)}) + \mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}(x)/m} 1_{\mathcal{V}_{t_n} > \mathcal{V}_{t_n}(x)}) = \mu_{t_n}^{(m)}(e^{[\mathcal{V}_{t_n} \wedge \mathcal{V}_{t_n}(x)]/m}) > 0.$$

Also notice that

$$\mu_{t_n}^{(m)} \left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+ \right) = \mu_{t_n}^{(m)}(\mathcal{V}_{t_n} > \mathcal{V}_{t_n}(x)) \times \left[\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m} \mid \mathcal{V}_{t_n} > \mathcal{V}_{t_n}(x)) - e^{\mathcal{V}_{t_n}(x)/m} \right]$$

so that a particle in some state x is more likely to be recycled when the potential values $\mathcal{V}_{t_n}(\bar{\mathcal{X}}_{t_n})$ of random states $\bar{\mathcal{X}}_{t_n}$ with law $\mu_{t_n}^{(m)}$ are more likely to be larger than $\mathcal{V}_{t_n}(x)$.

Finally, for bounded potential functions we observe that

$$\begin{aligned} & m \left[\mathcal{S}_{t_n, \mu_{t_n}^{(m)}}(f)(x) - f(x) \right] \\ &= m \frac{\mu_{t_n}^{(m)} \left((e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+ \right)}{\mu_{t_n}^{(m)}(e^{\mathcal{V}_{t_n}/m})} \left[\Psi_{(e^{\mathcal{V}_{t_n}/m} - e^{\mathcal{V}_{t_n}(x)/m})_+}(\mu_{t_n}^{(m)})(f) - f(x) \right] \\ &= \int [f(y) - f(x)] (\mathcal{V}_{t_n}(y) - \mathcal{V}_{t_n}(x))_+ \mu_{t_n}^{(m)}(dy) + O(1). \end{aligned}$$

Using some elementary calculations, we also prove a first order expansion of the same form as in (37).

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